Friedrichs Diagrams—Bosonic and Fermionic

Joint work with Morris Brooks
https://arxiv.org/abs/2303.13925

Sascha Lill
sascha.lill@unimi.it

Università degli Studi di Milano

June 19, 2025

Supported by ERC grant No. 101040991 "FermiMath"

Motivation

- Computing commutators $[A, B] = AB - BA$ is often necessary in many-body physics, and it causes a mess.

Motivation

- Computing commutators $[A, B] = AB - BA$ is often necessary in many-body physics, and it causes a mess.
- E.g., take bosonic **creation/annihilation operators** $a^*_{\boldsymbol{x}}, a_{\boldsymbol{y}}$ with $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{Z}^d$, satisfying the commutation relations:

$$[a_{\boldsymbol{y}}, a^*_{\boldsymbol{x}}] = \delta(\boldsymbol{x} - \boldsymbol{y}), \qquad [a_{\boldsymbol{y}}, a_{\boldsymbol{y}'}] = [a^*_{\boldsymbol{x}}, a^*_{\boldsymbol{x}'}] = 0.$$

Motivation

- Computing commutators $[A, B] = AB - BA$ is often necessary in many-body physics, and it causes a mess.
- E.g., take bosonic **creation/annihilation operators** $a^*_{\boldsymbol{x}}, a_{\boldsymbol{y}}$ with $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{Z}^d$, satisfying the commutation relations:

$$[a_{\boldsymbol{y}}, a^*_{\boldsymbol{x}}] = \delta(\boldsymbol{x} - \boldsymbol{y}), \qquad [a_{\boldsymbol{y}}, a_{\boldsymbol{y}'}] = [a^*_{\boldsymbol{x}}, a^*_{\boldsymbol{x}'}] = 0.$$

- Toy problem: $K := a^*_{\boldsymbol{x}} a_{\boldsymbol{y}}$ and $V := a^*_{\boldsymbol{x}'_2} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2}$. Compute:

$$[K, V] = a^*_{\boldsymbol{x}} a_{\boldsymbol{y}} a^*_{\boldsymbol{x}'_2} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} - a^*_{\boldsymbol{x}'_2} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} a^*_{\boldsymbol{x}} a_{\boldsymbol{y}}$$

- Task is considered solved if all operator products are **normal ordered**, i.e., of the form $a^* a^* \dots a^* a \dots aa$.

$$[K,V] = a^*_{\boldsymbol{x}} a_{\boldsymbol{y}} a^*_{\boldsymbol{x}'_2} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} - a^*_{\boldsymbol{x}'_2} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} a^*_{\boldsymbol{x}} a_{\boldsymbol{y}}$$

- Solution strategy: Apply CCR to normal-order the 2 operator products.

$$a^*_{\boldsymbol{x}} a_{\boldsymbol{y}} a^*_{\boldsymbol{x}'_2} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2}$$

$$[K,V] = a^*_{\boldsymbol{x}} a_{\boldsymbol{y}} a^*_{\boldsymbol{x}'_2} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} - a^*_{\boldsymbol{x}'_2} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} a^*_{\boldsymbol{x}} a_{\boldsymbol{y}}$$

- Solution strategy: Apply CCR to normal-order the 2 operator products.

$$\begin{aligned} & a^*_{\boldsymbol{x}} a_{\boldsymbol{y}} a^*_{\boldsymbol{x}'_2} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} \\ =& a^*_{\boldsymbol{x}} a_{\boldsymbol{y}} a^*_{\boldsymbol{x}'_2} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} - a^*_{\boldsymbol{x}} a^*_{\boldsymbol{x}'_2} a_{\boldsymbol{y}} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} + a^*_{\boldsymbol{x}} a^*_{\boldsymbol{x}'_2} a_{\boldsymbol{y}} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} \end{aligned}$$

$$[K,V] = a^*_{\boldsymbol{x}} a_{\boldsymbol{y}} a^*_{\boldsymbol{x}'_2} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} - a^*_{\boldsymbol{x}'_2} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} a^*_{\boldsymbol{x}} a_{\boldsymbol{y}}$$

- Solution strategy: Apply CCR to normal-order the 2 operator products.

$$\begin{aligned} & a^*_{\boldsymbol{x}} a_{\boldsymbol{y}} a^*_{\boldsymbol{x}'_2} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} \\ =& a^*_{\boldsymbol{x}} [a_{\boldsymbol{y}}, a^*_{\boldsymbol{x}'_2}] a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} + a^*_{\boldsymbol{x}} a^*_{\boldsymbol{x}'_2} a_{\boldsymbol{y}} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} \end{aligned}$$

$$[K,V] = a^*_{\boldsymbol{x}} a_{\boldsymbol{y}} a^*_{\boldsymbol{x}'_2} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} - a^*_{\boldsymbol{x}'_2} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} a^*_{\boldsymbol{x}} a_{\boldsymbol{y}}$$

- Solution strategy: Apply CCR to normal-order the 2 operator products.

$$\begin{aligned} & a^*_{\boldsymbol{x}} a_{\boldsymbol{y}} a^*_{\boldsymbol{x}'_2} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} \\ =& a^*_{\boldsymbol{x}} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} \delta(\boldsymbol{x}'_2 - \boldsymbol{y}) + a^*_{\boldsymbol{x}} a^*_{\boldsymbol{x}'_2} a_{\boldsymbol{y}} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} \end{aligned}$$

$$[K, V] = a^*_{\boldsymbol{x}} a_{\boldsymbol{y}} a^*_{\boldsymbol{x}'_2} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} - a^*_{\boldsymbol{x}'_2} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} a^*_{\boldsymbol{x}} a_{\boldsymbol{y}}$$

- Solution strategy: Apply CCR to normal-order the 2 operator products.

$$a^*_{\boldsymbol{x}} a_{\boldsymbol{y}} a^*_{\boldsymbol{x}'_2} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2}$$

$$= a^*_{\boldsymbol{x}} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} \delta(\boldsymbol{x}'_2 - \boldsymbol{y}) + a^*_{\boldsymbol{x}} a^*_{\boldsymbol{x}'_2} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} \delta(\boldsymbol{x}'_1 - \boldsymbol{y}) + a^*_{\boldsymbol{x}} a^*_{\boldsymbol{x}'_2} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2}$$

$$[K,V] = a^*_{\boldsymbol{x}} a_{\boldsymbol{y}} a^*_{\boldsymbol{x}'_2} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} - a^*_{\boldsymbol{x}'_2} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} a^*_{\boldsymbol{x}} a_{\boldsymbol{y}}$$

- Solution strategy: Apply CCR to normal-order the 2 operator products.

$$\begin{aligned} & a^*_{\boldsymbol{x}} a_{\boldsymbol{y}} a^*_{\boldsymbol{x}'_2} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} \\ = & a^*_{\boldsymbol{x}} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} \delta(\boldsymbol{x}'_2 - \boldsymbol{y}) + a^*_{\boldsymbol{x}} a^*_{\boldsymbol{x}'_2} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} \delta(\boldsymbol{x}'_1 - \boldsymbol{y}) + a^*_{\boldsymbol{x}} a^*_{\boldsymbol{x}'_2} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} \end{aligned}$$

$$\begin{aligned} & a^*_{\boldsymbol{x}'_1} a^*_{\boldsymbol{x}'_2} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} a^*_{\boldsymbol{x}} a_{\boldsymbol{y}} \\ = & a^*_{\boldsymbol{x}'_1} a^*_{\boldsymbol{x}'_2} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}} \delta(\boldsymbol{x} - \boldsymbol{y}'_2) + a^*_{\boldsymbol{x}'_1} a^*_{\boldsymbol{x}'_2} a_{\boldsymbol{y}'_2} a_{\boldsymbol{y}} \delta(\boldsymbol{x} - \boldsymbol{y}'_1) + a^*_{\boldsymbol{x}'_1} a^*_{\boldsymbol{x}'_2} a^*_{\boldsymbol{x}} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} a_{\boldsymbol{y}} \end{aligned}$$

$$[K,V] = a^*_{\boldsymbol{x}} a_{\boldsymbol{y}} a^*_{\boldsymbol{x}'_2} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} - a^*_{\boldsymbol{x}'_2} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} a^*_{\boldsymbol{x}} a_{\boldsymbol{y}}$$

- Solution strategy: Apply CCR to normal-order the 2 operator products.

$$\begin{aligned} & a^*_{\boldsymbol{x}} a_{\boldsymbol{y}} a^*_{\boldsymbol{x}'_2} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} \\ = & a^*_{\boldsymbol{x}} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} \delta(\boldsymbol{x}'_2 - \boldsymbol{y}) + a^*_{\boldsymbol{x}} a^*_{\boldsymbol{x}'_2} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} \delta(\boldsymbol{x}'_1 - \boldsymbol{y}) + a^*_{\boldsymbol{x}} a^*_{\boldsymbol{x}'_2} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} \end{aligned}$$

$$\begin{aligned} & a^*_{\boldsymbol{x}'_1} a^*_{\boldsymbol{x}'_2} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} a^*_{\boldsymbol{x}} a_{\boldsymbol{y}} \\ = & a^*_{\boldsymbol{x}'_1} a^*_{\boldsymbol{x}'_2} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}} \delta(\boldsymbol{x} - \boldsymbol{y}'_2) + a^*_{\boldsymbol{x}'_1} a^*_{\boldsymbol{x}'_2} a_{\boldsymbol{y}'_2} a_{\boldsymbol{y}} \delta(\boldsymbol{x} - \boldsymbol{y}'_1) + a^*_{\boldsymbol{x}} a^*_{\boldsymbol{x}'_1} a^*_{\boldsymbol{x}'_2} a_{\boldsymbol{y}} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} \end{aligned}$$

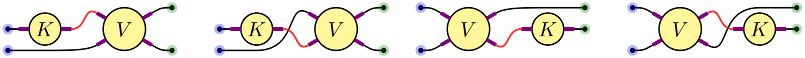
- Normal ordered terms cancel, 4 terms remain.
- Final result:

$$\begin{aligned} [K,V] = & a^*_{\boldsymbol{x}} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} \delta(\boldsymbol{x}'_2 - \boldsymbol{y}) + a^*_{\boldsymbol{x}} a^*_{\boldsymbol{x}'_2} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} \delta(\boldsymbol{x}'_1 - \boldsymbol{y}) \\ & - a^*_{\boldsymbol{x}'_1} a^*_{\boldsymbol{x}'_2} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}} \delta(\boldsymbol{x} - \boldsymbol{y}'_2) - a^*_{\boldsymbol{x}'_1} a^*_{\boldsymbol{x}'_2} a_{\boldsymbol{y}'_2} a_{\boldsymbol{y}} \delta(\boldsymbol{x} - \boldsymbol{y}'_1) \end{aligned}$$

- Final result:

$$[K,V] = a^*_{\boldsymbol{x}} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} \delta(\boldsymbol{x}'_2 - \boldsymbol{y}) + a^*_{\boldsymbol{x}} a^*_{\boldsymbol{x}'_2} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} \delta(\boldsymbol{x}'_1 - \boldsymbol{y})$$
$$- a^*_{\boldsymbol{x}'_1} a^*_{\boldsymbol{x}'_2} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}} \delta(\boldsymbol{x} - \boldsymbol{y}'_2) - a^*_{\boldsymbol{x}'_1} a^*_{\boldsymbol{x}'_2} a_{\boldsymbol{y}'_2} a_{\boldsymbol{y}} \delta(\boldsymbol{x} - \boldsymbol{y}'_1)$$

- Each term corresponds to a way how a^*-operators can "hop" over a-operators.
- Shortcut: Keep track of hoppings by diagrams.

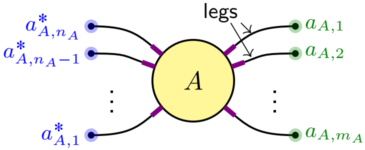


Mathematical Setting

- Particle coordinates: elements of measure space (X, μ)
- **1-particle Hilbert space**: $\mathfrak{h} := L^2(X, \mu)$
- **Fock space**: $\mathscr{F} := \bigoplus_{N=0}^{\infty} \mathfrak{h}^{\otimes N}$
- For $n_A, m_A \in \mathbb{N}$, $f_A \in L^2(X^{n_A} \times X^{m_A})$, $\boldsymbol{X}_A = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{n_A})$, $\boldsymbol{Y}_A = (\boldsymbol{y}_1, \ldots, \boldsymbol{y}_{m_A})$, the operator

$$A = \int f_A(\boldsymbol{X}_A, \boldsymbol{Y}_A) a^*_{A,n_A} \ldots a^*_{A,1} a_{A,1} \ldots a_{A,m_A} \, \mathrm{d}\boldsymbol{X}_A \mathrm{d}\boldsymbol{Y}_A$$

with $a^*_{\boldsymbol{x}_{A,j}} =: a^*_{A,j}$ and $a_{\boldsymbol{y}_{A,k}} =: a_{A,k}$, translates into:



The *-algebra of such operators A is called $\mathcal{A}_+$ (bosonic) or $\mathcal{A}_-$ (fermionic).

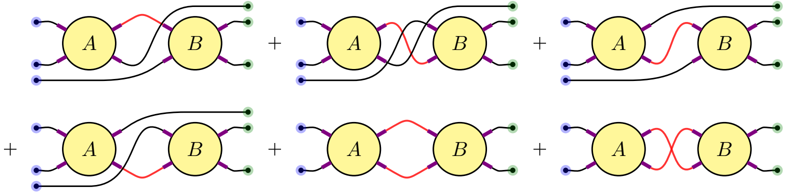
Attached Products

- For 2 diagrams A, B, the "sum over all possible contractions" is called **attached product**. For **bosons**,

$$A \multimap\!\!\circ\!\!- B := \sum_{(\pi,\pi')\in\mathcal{C}} \int f_A(\boldsymbol{X}_A, \boldsymbol{Y}_A) f_B(\boldsymbol{X}_B, \boldsymbol{Y}_B) \prod_{c=1}^{C} \delta(\boldsymbol{x}_{\pi(c)} - \boldsymbol{y}_{\pi'(c)}) \times$$

$$\times \left(\prod_{\ell=1}^{n_A} a_{A,\ell}\right)^* \left(\prod_{u\in\mathcal{U}_\pi} a_u\right)^* \prod_{u'\in\mathcal{U}'_{\pi'}} a_{u'} \prod_{\ell'=1}^{m_B} a_{B,\ell'} \, \mathrm{d}\boldsymbol{X}\mathrm{d}\boldsymbol{Y}$$

Here, we have $C \geqslant 1$ contractions, tracked by $\pi : \{1,\dots,C\} \to \{(B,1),\dots,(B,n_B)\}$ and $\pi' : \{1,\dots,C\} \to \{(A,1),\dots,(A,m_A)\}$.
$\mathcal{C}$ is the set of all admissible contractions.
$\mathcal{U}_\pi \subseteq \{(B,1),\dots,(B,n_B)\}$ and $\mathcal{U}'_{\pi'} \subseteq \{(A,1),\dots,(A,m_A)\}$ are uncontracted legs.



- Analogously, the **fermionic attached product** is

$$A \,{-}\!\!\circ\!\!{=}\, B := \sum_{(\pi,\pi')\in\mathcal{C}} \mathrm{sgn}(\pi,\pi') \int f_A(\boldsymbol{X}_A, \boldsymbol{Y}_A) f_B(\boldsymbol{X}_B, \boldsymbol{Y}_B) \prod_{c=1}^{C} \delta(\boldsymbol{x}_{\pi(c)} - \boldsymbol{y}_{\pi'(c)}) \times$$

$$\times \left(\prod_{\ell=1}^{n_A} a_{A,\ell} \right)^* \left(\prod_{u\in\mathcal{U}} a_u \right)^* \prod_{u'\in\mathcal{U}'} a_{u'} \prod_{\ell'=1}^{m_B} a_{B,\ell'} \,\mathrm{d}\boldsymbol{X}\,\mathrm{d}\boldsymbol{Y}$$

Here, $\mathrm{sgn}(\pi,\pi') \in \{1,-1\}$ is some sign factor.

Main Result: Commutator Formulas

Theorem (Bosonic Commutator Formula)

Consider $A, B \in \mathcal{A}_+$, i.e., the CCR hold. Then,

$$[A, B] = A\!-\!\circ\!-\!B - B\!-\!\circ\!-\!A$$

Theorem (Fermionic Commutator Formula, [Brooks, L. 2023])

Consider $A, B \in \mathcal{A}_-$, i.e., the CAR hold. Then,

$$[A, B] = A\!-\!\circ\!=\!B - B\!-\!\circ\!=\!A \qquad \text{if } (n_A + m_A)(n_B + m_B) \text{ is even,}$$
$$\{A, B\} = A\!-\!\circ\!=\!B + B\!-\!\circ\!=\!A \qquad \text{if } (n_A + m_A)(n_B + m_B) \text{ is odd.}$$

- Proof idea: By induction, similar to proving Wick's theorem.

The Fermionic Sign Factor

- What is $\mathrm{sgn}(\pi, \pi')$?
- For fermions, $a_{\boldsymbol{y}} a^*_{\boldsymbol{x}} + a^*_{\boldsymbol{x}} a_{\boldsymbol{y}} = \delta(\boldsymbol{x} - \boldsymbol{y}) \Leftrightarrow a_{\boldsymbol{y}} a^*_{\boldsymbol{x}} = -a^*_{\boldsymbol{x}} a_{\boldsymbol{y}} + \delta(\boldsymbol{x} - \boldsymbol{y})$
  $\Rightarrow$ We pick up a (-1), whenever we hop. E.g.,

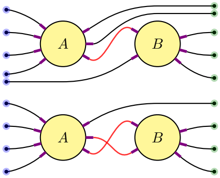
$$
\begin{aligned}
&+ a^*_{\boldsymbol{x}_4} a^*_{\boldsymbol{x}_3} a^*_{\boldsymbol{x}_2} a^*_{\boldsymbol{x}_1} a_{\boldsymbol{y}_1} a_{\boldsymbol{y}_2} a_{\boldsymbol{y}_3} a^*_{\boldsymbol{x}'_2} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} a_{\boldsymbol{y}'_3} \\
=& - a^*_{\boldsymbol{x}_4} a^*_{\boldsymbol{x}_3} a^*_{\boldsymbol{x}_2} a^*_{\boldsymbol{x}_1} a_{\boldsymbol{y}_1} a_{\boldsymbol{y}_2} a^*_{\boldsymbol{x}'_2} a_{\boldsymbol{y}_3} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} a_{\boldsymbol{y}'_3} + \dots \\
=& + a^*_{\boldsymbol{x}_4} a^*_{\boldsymbol{x}_3} a^*_{\boldsymbol{x}_2} a^*_{\boldsymbol{x}_1} a_{\boldsymbol{y}_1} a^*_{\boldsymbol{x}'_2} a_{\boldsymbol{y}_2} a_{\boldsymbol{y}_3} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} a_{\boldsymbol{y}'_3} + \dots
\end{aligned}
$$

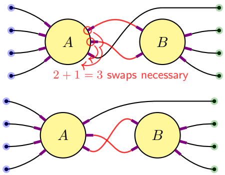
The Fermionic Sign Factor

- What is $\mathrm{sgn}(\pi, \pi')$?
- For fermions, $a_{\boldsymbol{y}} a^*_{\boldsymbol{x}} + a^*_{\boldsymbol{x}} a_{\boldsymbol{y}} = \delta(\boldsymbol{x} - \boldsymbol{y}) \Leftrightarrow a_{\boldsymbol{y}} a^*_{\boldsymbol{x}} = -a^*_{\boldsymbol{x}} a_{\boldsymbol{y}} + \delta(\boldsymbol{x} - \boldsymbol{y})$
  $\Rightarrow$ We pick up a (-1), whenever we hop. E.g.,

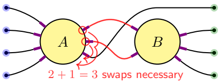
$$
\begin{aligned}
&+ a^*_{\boldsymbol{x}_4} a^*_{\boldsymbol{x}_3} a^*_{\boldsymbol{x}_2} a^*_{\boldsymbol{x}_1} a_{\boldsymbol{y}_1} a_{\boldsymbol{y}_2} a_{\boldsymbol{y}_3} a^*_{\boldsymbol{x}'_2} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} a_{\boldsymbol{y}'_3} \\
= &- a^*_{\boldsymbol{x}_4} a^*_{\boldsymbol{x}_3} a^*_{\boldsymbol{x}_2} a^*_{\boldsymbol{x}_1} a_{\boldsymbol{y}_1} a_{\boldsymbol{y}_2} a^*_{\boldsymbol{x}'_2} a_{\boldsymbol{y}_3} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} a_{\boldsymbol{y}'_3} + \dots \\
= &+ a^*_{\boldsymbol{x}_4} a^*_{\boldsymbol{x}_3} a^*_{\boldsymbol{x}_2} a^*_{\boldsymbol{x}_1} a_{\boldsymbol{y}_1} a^*_{\boldsymbol{x}'_2} a_{\boldsymbol{y}_2} a_{\boldsymbol{y}_3} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} a_{\boldsymbol{y}'_3} + \dots
\end{aligned}
$$

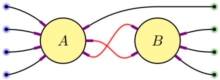
- Contraction of first hopping always has "+":
  $+\delta(\boldsymbol{x}'_2 - \boldsymbol{y}_3) a^*_{\boldsymbol{x}_4} a^*_{\boldsymbol{x}_3} a^*_{\boldsymbol{x}_2} a^*_{\boldsymbol{x}_1} a_{\boldsymbol{y}_1} a_{\boldsymbol{y}_2} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} a_{\boldsymbol{y}'_3}$,
  $+\delta(\boldsymbol{x}'_2 - \boldsymbol{y}_3)\delta(\boldsymbol{x}'_1 - \boldsymbol{y}_2) a^*_{\boldsymbol{x}_4} a^*_{\boldsymbol{x}_3} a^*_{\boldsymbol{x}_2} a^*_{\boldsymbol{x}_1} a_{\boldsymbol{y}_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} a_{\boldsymbol{y}'_3}$,
- We call these diagrams "maximally crossed".



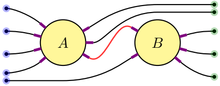


- We can make any diagram maximally crossed by index permutations σ and σ' of $\{1, \dots, m_A\}$ and $\{1, \dots, n_B\}$
- CAR imply $a_{\boldsymbol{y}} a_{\boldsymbol{y}'} = -a_{\boldsymbol{y}'} a_{\boldsymbol{y}}$, $a^*_{\boldsymbol{x}} a^*_{\boldsymbol{x}'} = -a^*_{\boldsymbol{x}'} a^*_{\boldsymbol{x}}$, so each swap in σ, σ' picks up a (-1).
- All swaps together yield a factor of $\mathrm{sgn}(\sigma)\mathrm{sgn}(\sigma')$.



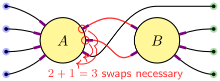


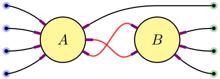
- We can make any diagram maximally crossed by index permutations σ and σ' of $\{1, \ldots, m_A\}$ and $\{1, \ldots, n_B\}$
- CAR imply $a_{\boldsymbol{y}} a_{\boldsymbol{y}'} = -a_{\boldsymbol{y}'} a_{\boldsymbol{y}}$, $a^*_{\boldsymbol{x}} a^*_{\boldsymbol{x}'} = -a^*_{\boldsymbol{x}'} a^*_{\boldsymbol{x}}$, so each swap in σ, σ' picks up a (-1).
- All swaps together yield a factor of $\mathrm{sgn}(\sigma)\mathrm{sgn}(\sigma')$.



- Finally, we must normal order diagrams like $\delta(\boldsymbol{x}'_2 - \boldsymbol{y}_3) a^*_{\boldsymbol{x}_4} a^*_{\boldsymbol{x}_3} a^*_{\boldsymbol{x}_2} a^*_{\boldsymbol{x}_1} a_{\boldsymbol{y}_1} a_{\boldsymbol{y}_2} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} a_{\boldsymbol{y}'_3}$:
- After C contractions, it remains to pull $(m_A - C)$ operators a past $(n_B - C)$ operators a^* $\Rightarrow$ factor of $(-1)^{(m_A - C)(n_B - C)}$

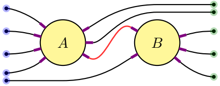
- We can make any diagram maximally crossed by index permutations σ and σ' of $\{1, \dots, m_A\}$ and $\{1, \dots, n_B\}$





- CAR imply $a_{\boldsymbol{y}} a_{\boldsymbol{y}'} = -a_{\boldsymbol{y}'} a_{\boldsymbol{y}}$, $a^*_{\boldsymbol{x}} a^*_{\boldsymbol{x}'} = -a^*_{\boldsymbol{x}'} a^*_{\boldsymbol{x}}$, so each swap in σ, σ' picks up a (-1).
- All swaps together yield a factor of $\mathrm{sgn}(\sigma)\mathrm{sgn}(\sigma')$.

- Finally, we must normal order diagrams like $\delta(\boldsymbol{x}'_2 - \boldsymbol{y}_3) a^*_{\boldsymbol{x}_4} a^*_{\boldsymbol{x}_3} a^*_{\boldsymbol{x}_2} a^*_{\boldsymbol{x}_1} a_{\boldsymbol{y}_1} a_{\boldsymbol{y}_2} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2} a_{\boldsymbol{y}'_3}$:



- After C contractions, it remains to pull $(m_A - C)$ operators a past $(n_B - C)$ operators a^* $\Rightarrow$ factor of $(-1)^{(m_A - C)(n_B - C)}$

- Final result:

$$\mathrm{sgn}(\pi, \pi') = (-1)^{(m_A - C)(n_B - C)} \mathrm{sgn}(\sigma)\mathrm{sgn}(\sigma')$$

Application: Hartree Equation

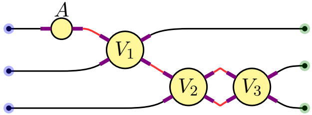
- Consider very generic bosonic many-body Hamiltonian:

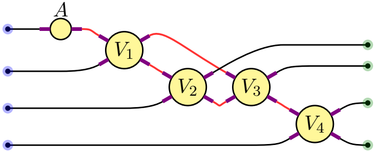
$$H = K + V := \sum_{\boldsymbol{x},\boldsymbol{y}} f_K(\boldsymbol{x},\boldsymbol{y}) a^*_{\boldsymbol{x}} a_{\boldsymbol{y}} + \sum_{\boldsymbol{x}'_1,\boldsymbol{x}'_2,\boldsymbol{y}'_1,\boldsymbol{y}'_2} f_V(\boldsymbol{x}'_1,\boldsymbol{x}'_2,\boldsymbol{y}'_1,\boldsymbol{y}'_2) a^*_{\boldsymbol{x}'_2} a^*_{\boldsymbol{x}'_1} a_{\boldsymbol{y}'_1} a_{\boldsymbol{y}'_2}$$

- Interaction picture dynamics: $\mathcal{U}_t = e^{-itH} e^{itK} : \mathscr{F} \to \mathscr{F}$.
- Time evolution of observable $A \in \mathcal{A}_+$ via Dyson series:

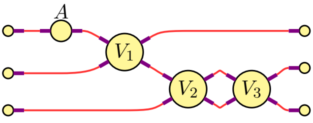
$$A_t := \mathcal{U}_t^{-1} A \mathcal{U}_t = \sum_{k=0}^{\infty} \int_{0 \leqslant t_1 \leqslant \cdots \leqslant t_k \leqslant t} \underbrace{\frac{1}{i^k} \left[\ldots \left[A, \frac{1}{2} V_1 \right], \ldots, \frac{1}{2} V_k \right]}_{=: A_k^{t_1,\ldots,t_k}} \mathrm{d}t_1 \ldots \mathrm{d}t_k$$

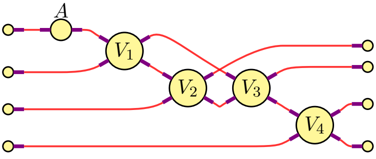
with $V_n := e^{-it_n K} V e^{it_n K}$.





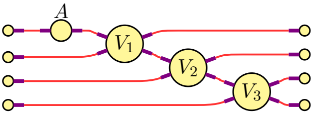
- If system is initially in coherent state (BEC) over $u_0 \in \mathfrak{h}$, then expectation of A at time t is $\langle f_{A_t} \rangle_{e^{-itK_1} u_0}$ with
  - 1-body kinetic operator $(K_1 u)(\boldsymbol{x}) := \sum_{\boldsymbol{y}} f_K(\boldsymbol{x}, \boldsymbol{y}) u(\boldsymbol{y})$
  - $\langle f_A \rangle_u := \sum_{\boldsymbol{X},\boldsymbol{Y}} f_A(\boldsymbol{X}, \boldsymbol{Y}) \overline{u(\boldsymbol{x}_n)} \dots \overline{u(\boldsymbol{x}_1)} u(\boldsymbol{y}_1) \dots u(\boldsymbol{y}_m)$

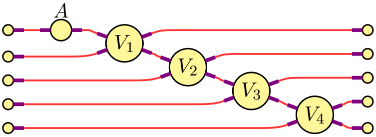




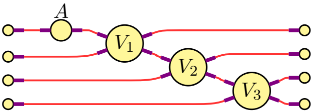
- One can approximate $\langle f_{A_t} \rangle_{e^{-itK_1} u_0}$ by $\langle f_A \rangle_{u_t}$ with u_t solving **Hartree equation** $\boxed{i\partial_t u_t = (K_1 + V_{u_t}) u_t}$ with:
  - initial data u_0 at $t = 0$
  - $(V_{u'} u)(\boldsymbol{x}) := \sum_{\boldsymbol{x}_2, \boldsymbol{y}_1, \boldsymbol{y}_2} f_V(\boldsymbol{x}, \boldsymbol{x}_2, \boldsymbol{y}_1, \boldsymbol{y}_2) u'(\boldsymbol{x}_2) u'(\boldsymbol{y}_1) u(\boldsymbol{y}_2)$

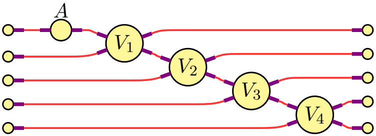
- Hartree prediction $\langle f_A \rangle_{u_t}$ arises from subset of diagrams:
- Define $f_{k,k+1}^{t_1,\ldots,t_k}$ as the sum of all diagrams in $A_k^{t_1,\ldots,t_k}$ with $k+1$ legs.





- Hartree prediction $\langle f_A \rangle_{u_t}$ arises from subset of diagrams:
- Define $f_{k,k+1}^{t_1,\ldots,t_k}$ as the sum of all diagrams in $A_k^{t_1,\ldots,t_k}$ with $k+1$ legs.





Proposition ([Brooks, L. 2023] Hartree Dynamics in Diagrams)

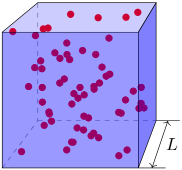
Let $T := \frac{1}{2\|V_2\|\|u_0\|^2}$ and $(u_t)_{t\in(-T,T)}$ solve the Hartree equation. Then, for all $|t| < T$, we have

$$\langle f_A \rangle_{u_t} = \sum_{k=0}^{\infty} \int_{0\leqslant t_1\leqslant\cdots\leqslant t_k\leqslant t} \langle f_{k,k+1}^{t_1,\ldots,t_k} \rangle_{e^{-itK_1}u_0} \, \mathrm{d}t_1 \ldots \mathrm{d}t_k,$$

where $V_2 : \mathfrak{h}\otimes\mathfrak{h} \to \mathfrak{h}\otimes\mathfrak{h}$ is the operator associated with the kernel f_V.

Application: Fermi Gases

- N fermions on a torus $[0, 2\pi]^3$
- Hilbert space: $\mathscr{H}^{(N)} := L^2([0, 2\pi]^3)^{\otimes_{\mathrm{a}} N}$
  That means, $\psi \in \mathscr{H}^{(N)}$ is antisymmetric
  $\psi(\ldots, x_i, \ldots, x_j, \ldots) = -\psi(\ldots, x_j, \ldots, x_i, \ldots).$



- Hamiltonian $H_N : \mathscr{H}^{(N)} \supset \mathrm{dom}(H_N) \to \mathscr{H}^{(N)}$,

$$H_N := \sum_{j=1}^{N} -\Delta_{x_j} + N^{-\frac{1}{3}} \sum_{i<j}^{N} V(x_i - x_j)$$

- We now write this using a^*, a-operators.

Application to Fermi Gases

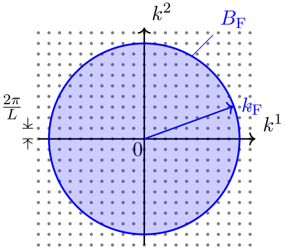
- Fock space is $\mathscr{F} := \bigoplus_{N=0}^{\infty} \mathscr{H}^{(N)}$ with vacuum vector $\Omega = (1, 0, 0, \dots) \in \mathscr{F}$
- Define a_p^*, a_p on $\mathscr{F}$, which create/annihilate plane waves $e_p \in L^2([0, 2\pi])$ with $e_p(x) := \frac{1}{(2\pi)^{3/2}} e^{ip \cdot x}$ and $p \in \mathbb{Z}^3$
- Then, lifting H_N from $\mathscr{H}^{(N)}$ to $\mathscr{F}$ yields

$$\mathcal{H}_N = \sum_{p \in \mathbb{Z}^3} |p|^2 a_p^* a_p + \frac{1}{2} N^{-\frac{1}{3}} \sum_{k,p,q \in \mathbb{Z}^3} \hat{V}(k) a_{p+k}^* a_{q-k}^* a_q a_p$$

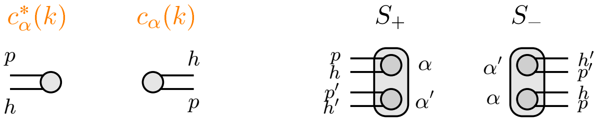
("lifting" means $\mathcal{H}_N := \bigoplus_{N=0}^{\infty} H_N$)
(We will switch between H_N and $\mathcal{H}_N$, or $\mathscr{H}^{(N)}$ and $\mathscr{F}$, as needed.)

- Physically, system will be approximately in the **ground state**, that is, the lowest eigenvector of H_N
- First approximation to GS: fill up the Fermi ball $B_{\mathrm{F}} := B_{k_{\mathrm{F}}}(0)$, s.t. $|B_{\mathrm{F}}| = N$
- **Fermi ball state** is $\psi_{\mathrm{FB}} := R\Omega \in \mathscr{H}^{(N)} \subset \mathscr{F}$ with **particle–hole transformation** $R = R^* = R^{-1} : \mathscr{F} \to \mathscr{F}$ defined via

$$Ra_pR := \begin{cases} a_p & \text{if } p \notin B_{\mathrm{F}} \\ a_p^* & \text{if } p \in B_{\mathrm{F}} \end{cases}$$

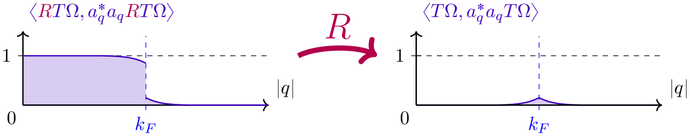


- Even better approximation [Benedikter, Nam, Porta, Schlein, Seiringer 2021+]: $\boxed{\psi = RT\Omega}$ with
  $T : \mathscr{F} \to \mathscr{F}$: Almost-Bogoliubov transformation, $T^* = T^{-1}$
  $$T := e^{-S} , \quad S := \sum_{k,\alpha,\beta} K(k)_{\alpha,\beta} \, c^*_\alpha(k) c^*_\beta(k) - \text{h.c.} =: S_+ + S_- ,$$
  where k, α, β are indices, $K(k)_{\alpha,\beta} \in \mathbb{R}$
- Here, $c^*_\alpha(k) = \sum_{p,h} f(p,h) a^*_p a^*_h$ with $p \notin B_{\mathrm{F}}$ (particle) and $h \in B_{\mathrm{F}}$ (hole). So **particles do not contract with holes**
- $[c_\alpha(k), c^*_{\alpha'}(k')] = \delta_{\alpha,\alpha'}(\delta_{k,k'} + \mathcal{E}_\alpha(k,k'))$ ("almost-CCR"), so $c_\alpha(k)$ are **almost bosonic**
- As Friedrichs diagrams:



- Within ψ, we compute the **momentum distribution** $\boxed{q \mapsto \langle \psi, a_q^* a_q \psi \rangle}$, $q \in \mathbb{Z}^3$.

- Action of R in $\psi = Re^{-S}\Omega$ is easily described:

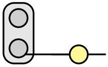


- Equivalently, compute **excitation density**

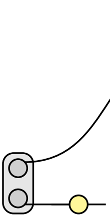
$$\langle n_q \rangle := \langle \Omega, e^S a_q^* a_q e^{-S} \Omega \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \Omega, \mathrm{ad}_S^n (a_q^* a_q) \Omega \rangle$$

with $\mathrm{ad}_A^n(B) := [A, \ldots [A, [A, B]] \ldots]$: n-fold multicommutator

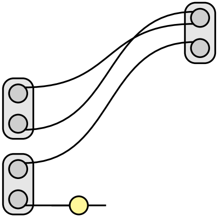
- Multicommutator computation: start with vertex of $a_q^* a_q$



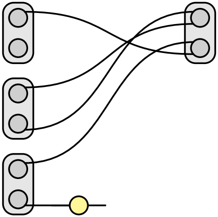
- Multicommutator computation: start with vertex of $a_q^* a_q$
- Consecutively add vertices $S_\pm$, contracting $\geqslant 1$ legs per step



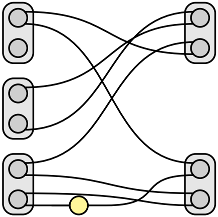
- Multicommutator computation: start with vertex of $a_q^* a_q$
- Consecutively add vertices $S_\pm$, contracting $\geqslant 1$ legs per step



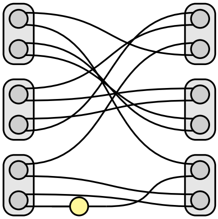
- Multicommutator computation: start with vertex of $a_q^* a_q$
- Consecutively add vertices $S_\pm$, contracting $\geqslant 1$ legs per step



- Multicommutator computation: start with vertex of $a_q^* a_q$
- Consecutively add vertices $S_\pm$, contracting $\geqslant 1$ legs per step

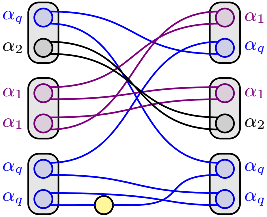


- Multicommutator computation: start with vertex of $a_q^* a_q$
- Consecutively add vertices $S_\pm$, contracting $\geqslant 1$ legs per step

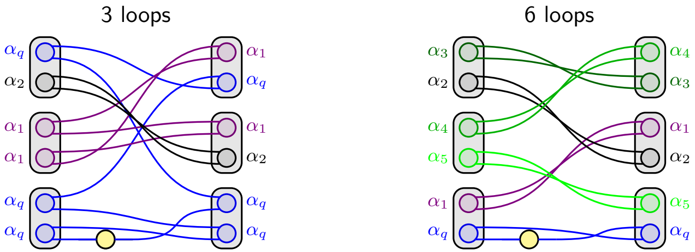


- Multicommutator computation: start with vertex of $a_q^* a_q$
- Consecutively add vertices $S_\pm$, contracting $\geqslant 1$ legs per step
- In the end, taking $\langle \Omega, \, \cdot \, \Omega \rangle$ removes all diagrams with external legs, since $a_p \Omega = 0$.

3 loops



- Multicommutator computation: start with vertex of $a_q^* a_q$
- Consecutively add vertices $S_\pm$, contracting $\geqslant 1$ legs per step
- In the end, taking $\langle \Omega, \cdot\, \Omega \rangle$ removes all diagrams with external legs, since $a_p \Omega = 0$.
- In each "loop", indices α_j are fixed. The final result has a sum $\sum_{\alpha_j}$ for each loop



- Multicommutator computation: start with vertex of $a_q^* a_q$
- Consecutively add vertices $S_\pm$, contracting $\geqslant 1$ legs per step
- In the end, taking $\langle \Omega, \cdot \, \Omega \rangle$ removes all diagrams with external legs, since $a_p \Omega = 0$.
- In each "loop", indices α_j are fixed. The final result has a sum $\sum_{\alpha_j}$ for each loop
- Biggest contribution is with most loops → 2 fermions together act as a single boson → **bosonized graph**

Theorem ([L. 2024])

$$\langle n_q \rangle = \sum_{\substack{n=2 \\ n:\text{even}}}^{\infty} \sum_{\substack{G:\text{graph with} \\ n \text{ vertices}}} \mathsf{Val}(G)$$

- Summing up bosonized graphs gives $\langle n_q \rangle \approx N^{-\frac{2}{3}} I(q)$, agreeing with conjecture by [Daniel, Vosko 1960]
- Further diagrams are $\mathcal{O}(N^{-1})$, but absolute convergence is not established, yet.
- Nevertheless, with many-body analysis, we proved:

Theorem ([Benedikter, L. 2023])

For $\hat{V} \geqslant 0$ compactly supported and most $q \in \mathbb{Z}^3$,

$$\boxed{\langle n_q \rangle = N^{-\frac{2}{3}} I(q) + \mathcal{O}(N^{-\frac{2}{3}-\frac{1}{12}})},$$

Thank you for your attention!