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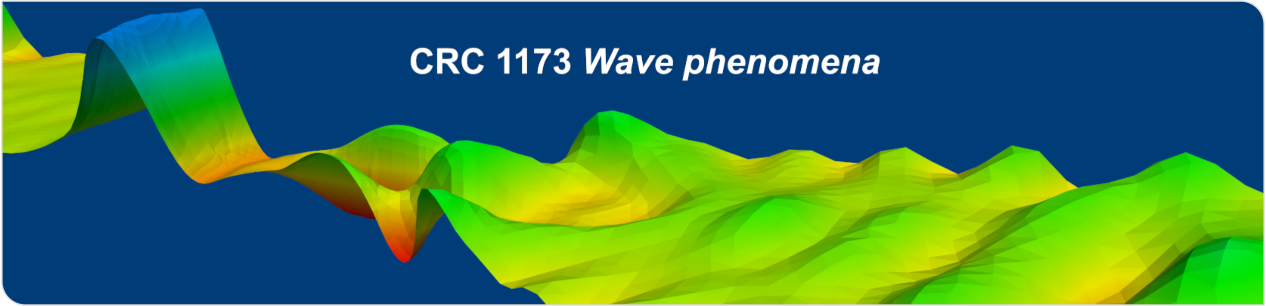
Derivation of the Effective Dynamics for the Bose Polaron

Jonas Lampart, Peter Pickl, Siegfried Spruck | 19. June 2025



Plan

Motivation: Impurity Particle

Quantum gas of N bosons with 1 impurity particle $\rightarrow$ Tracer particle.

Applications of **impurity** systems in physics:

- **Track local structure**:
  Vortex lattice in liquid Helium.
- Local **probing** the **density distribution** of a Bose gas [Schmid-Härter-Denschlag 10].

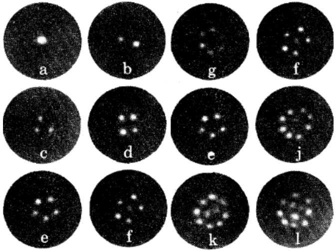


Figure: Rotating Helium with marked vortices by tracer particles [Varmchuk-Gordon-Packard 79].

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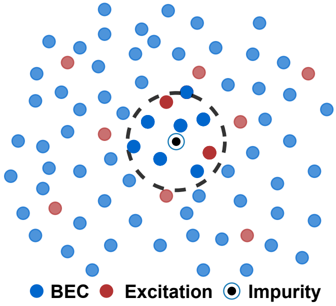

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Bose Polaron: Quasi-particle of impurity and bosons.

Goal: Verify Bose Polaron dynamics for our system.
$\rightarrow$ Effective dynamics.

1. Motivation: Impurity Particles in Experiments

2. System Set-Up: Bose Gas in Condensation

3. Main Theorem: Validity Bose Polaron Dynamics

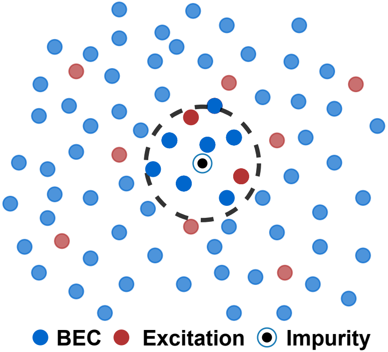
4. Methods: Condensate Control & Impurity Localization

System Set-Up

- Quantum gas of N bosons, 1 impurity particle in $\mathbb{R}^3$.
- Initial **volume** Λ, **density** $\rho = \frac{N}{\Lambda}$ with ρ, Λ **large**.
- **Interactions** $V, W \in \mathscr{S}(\mathbb{R}^3, \mathbb{R})$ **weak**, even and of range $\mathcal{O}(1)$: **Mean-field scaling**.
- Dynamics on $L^2(\mathbb{R}^3_x) \otimes L^2_{\text{sym}}(\mathbb{R}^{3N}_y)$

$$
\begin{aligned}
\mathrm{i}\partial_t \psi_{N,t} &= H_N \psi_{N,t}\,, \\
H_N &= -\sum_{i=1}^{N} \frac{\Delta_{y_i}}{2} - \frac{\Delta_x}{2m} + \frac{1}{\rho} \sum_{1 \le i < j \le N} V(y_i - y_j) \\
&\quad + \frac{1}{\sqrt{\rho}} \sum_{i=1}^{N} W(x - y_i)\,.
\end{aligned}
$$

x: Impurity position; y_i: Boson positions.



Bose-Einstein Condensate

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Complete **Bose-Einstein condensation**
if almost all bosons in same state:

$$\psi_{N,t}(y_1,\ldots,y_N) \sim \prod_{i=1}^{N} \varphi_t(y_i)\,,$$

for large ρ, $\varphi_t \in L^2(\mathbb{R}^3)$ a **one-particle state**.

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Time evolution of **condensate** φ_t:

$$i\partial_t\varphi_t(y) = \Big(-\frac{\Delta}{2} + V * |\varphi_t|^2(y) - \underbrace{\mu_t}_{\in\mathbb{R}}\Big)\varphi_t(y) \quad \textbf{(Hartree-eq)}\,,$$

$$\varphi_{t=0} = \varphi_0\,.$$

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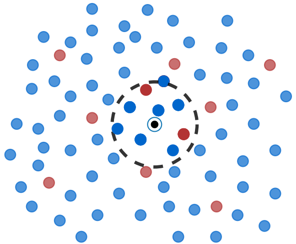

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The **condensate** φ_t defines a **background/environment** for the excitation **dynamics** we are actually interest in.

Excitations

We call $\zeta \in \{\varphi_t\}^\perp \subset L^2(\mathbb{R}^3) = \text{lin}\{\varphi_t\} \oplus \{\varphi_t\}^\perp$ an **excitation** out of the condensate. These excitations can emerge and disappear. Define U_t, **isometry**, mapping into the **excitation space** $\mathcal{F}(\{\varphi_t\}^\perp)$

$$U_t : L^2_{\text{sym}}(\mathbb{R}^{3N}) \to \mathcal{F}(\{\varphi_t\}^\perp) = \bigoplus_{k=0}^{\infty} (\{\varphi_t\}^\perp)^{\otimes_s k}$$

$$\underbrace{\varphi_t \otimes_s \cdots \otimes_s \varphi_t}_{(N-k)\text{ times}} \otimes_s \underbrace{\zeta_1 \otimes_s \ldots \zeta_k}_{\in (\{\varphi_t\}^\perp)^{\otimes_s k}} \xmapsto{U_t} \zeta_1 \otimes_s \ldots \zeta_k \,.$$

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$U_t\psi_{N,t}$: excitation part of the wave function evolved by

$$\begin{aligned} \mathrm{i}\partial_t U_t\psi_{N,t} &= H^{\mathrm{ex}} U_t\psi_{N,t} \,, \\ H^{\mathrm{ex}} &= U_t H_N U_t^* + i(\partial_t U_t) U_t^* \quad \textbf{(Excitation Hamiltonian)} \,. \end{aligned}$$

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Our setting: System exhibits **Bose-Einstein condensation** with **few excitations**.

Bose Polaron

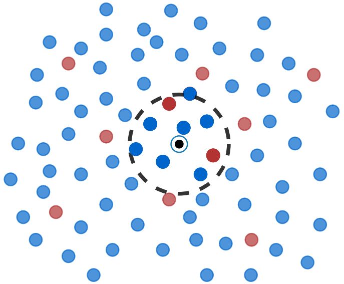
Bose Polaron: Quasi-particle of impurity and bosons.

It is described by effective dynamics generated by **Bogoliubov-Fröhlich** Hamiltonian H^{BF}

$$H^{\mathrm{BF}} = H^{\mathrm{Bog}} - \frac{\Delta_x}{2m} + a(Q_t W_x \varphi_t) + a^*(Q_t W_x \varphi_t)\,,$$

$$\mathrm{i}\partial_t \psi_t^{\mathrm{BF}} = H^{\mathrm{BF}} \psi_t^{\mathrm{BF}}\,,$$

on $L^2(\mathbb{R}^3_x) \otimes \mathcal{F}(\{\varphi_t\}^\perp) =: \mathcal{F}^{\mathrm{ex}}$.



$H^{\mathrm{Bog}} \sim d\Gamma(\xi)$ Bogoliubov Hamiltonian, modelling free excitations. Q_t projects into $\{\varphi_t\}^\perp$. $a^\#(Q_t W_x \varphi_t)$ **creates or annihilates excitation** due to interaction of impurity with condensate.

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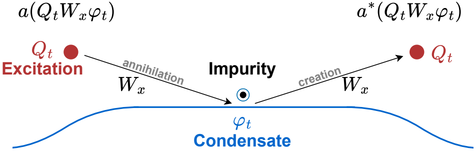

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Importance of the **Bogoliubov-Fröhlich** Hamiltonian H^{BF}:

- Its validity indicates the **formation** of the **Bose Polaron**.
- We started with Schrödinger's equation and now consider a QFT of **matter** (impurity) interacting with a **field of excitations**.
- It **simplifies** the **dynamics**: "Free" quantum field interacting with matter.

Literature

Derivation of the Bogoliubov-Fröhlich Hamiltonian from the microscopic dynamics:

Myśliwy-Seiringer 2020: **Spectrum** at low energies on the unit torus in the **mean-field scaling** (Interactions are often but weak).

Lampart-Pickl 2022: **Dynamics** on the unit torus in the **mean-field scaling**.

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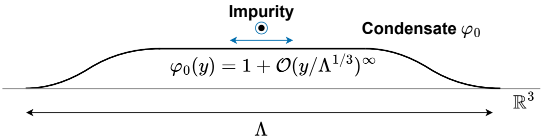
Our work extents the results of **[Lampart-Pickl 2022]** to **large volumes** and without periodic boundary condition leading to a **non-constant condensate**. We show the **validity** of the **Bose Polaron dynamics** in the limit of **high densities** $\rho >> 1$ and **large volumes** $\Lambda >> 1$

$$\underbrace{H^{\mathrm{ex}}}_{\text{Full excitation dynamics}} \xrightarrow{\Lambda,\rho\to\infty} \underbrace{H^{\mathrm{BF}}}_{\text{Polaron dynamics}}.$$



Initial conditions

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2. System Set-Up: Bose Gas in Condensation

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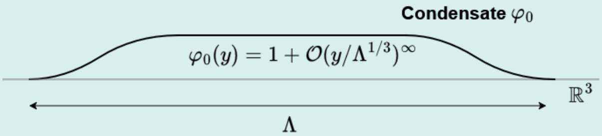
Main Theorem: Validity Bose Polaron Dynamics

Theorem (Lampart, Pickl, S.)

If we have the initial conditions:

i) ***Condensate** is a **rescaled function** $\varphi_0(y) = \eta(\Lambda^{-\frac{1}{3}} y)$, η independent of ρ, Λ and φ_0 **flat around the origin**.*



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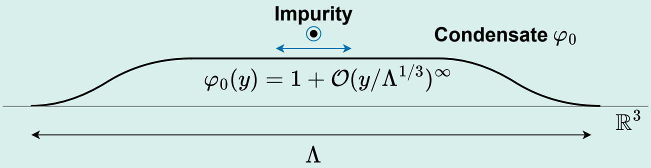

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ii) ***Excitation number≤Volume****:* $\left\langle \psi_0^{\mathrm{BF}}, (\mathcal{N}_+ + 1)^n \psi_0^{\mathrm{BF}} \right\rangle \leq C_n \Lambda^n$, $0 \leq n \leq 4$.

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iii) ***Impurity-localized** around origin:*
$\left\langle \psi_0^{\mathrm{BF}}, (-\Delta_x + x^2 + U_{\mathcal{V}}(\mathcal{N}_+ + 1)U_{\mathcal{V}}^*)^M \psi_0^{\mathrm{BF}} \right\rangle \leq C_M$, *for all $M \geq 0$.*

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Then for $\psi_0^{\mathrm{ex}} = \psi_0^{\mathrm{BF}}$ and for all times $T \geq 0$ there exists a constant $C > 0$ such that for all densities $\rho \geq 1$, volumes $\Lambda \geq 1$

$$\sup_{t \in [-T,T]} \| \underbrace{e^{it(\rho^{1/2} \int W - \mu_0)} \psi_t^{\mathrm{ex}}}_{\text{Full excitation dynamics}} - \underbrace{\psi_t^{\mathrm{BF}}}_{\text{Polaron dynamics}} \|_{\mathcal{F}^{\mathrm{ex}}} \leq C \left(\frac{\Lambda^3}{\rho} \right)^{1/2} .$$

*Convergence to 0 for $\rho, \Lambda \to \infty$, with $\Lambda^3 \ll \rho$. We conclude **effective description** of the **full microscopic dynamics** through $\mathbf{H}^{\mathrm{BF}}$.*

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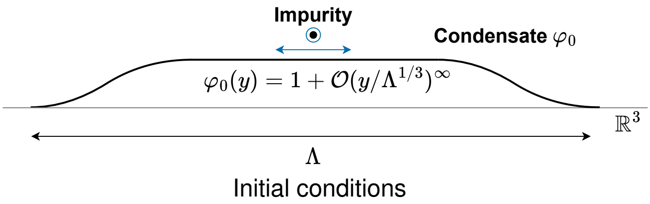
Methods Overview

$$\underbrace{H^{\mathrm{ex}}}_{\text{Full excitation dynamics}} = \underbrace{H^{\mathrm{BF}}}_{\text{Polaron dynamics}} + 1/\sqrt{\rho} \cdot \text{error} + \underbrace{\sqrt{\rho} W * |\varphi_t|^2(x)}_{\text{Mean-field impurity-condensate interaction}} .$$

We want to show

$$H^{\mathrm{ex}} \overset{\Lambda,\rho\to\infty}{\longrightarrow} H^{\mathrm{BF}} .$$

- Control **excitation number** similar to [Petrat-Pickl-Soffer 20, Lewin-Nam-Schlein 15]
  $\rightarrow \left(1/\sqrt{\rho} \cdot \text{error}\right)$ small.
- Control **impurity localization** and φ_t remains **flat around the origin**
  $\rightarrow \sqrt{\rho} W * |\varphi_t|^2(x) \sim \sqrt{\rho} W * 1$.



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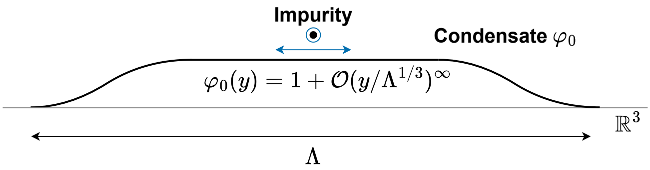


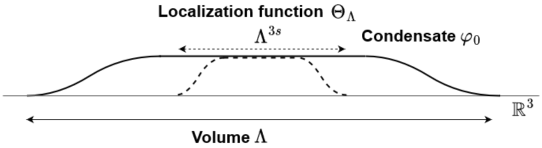
Initial conditions

Condensate Control

Condensate Control
Show: Condensate remains flat around origin.

↓

Propagation estimates for $\|\Theta_\Lambda(\varphi_t - \tilde{\varphi}_t)\|_2$, with localization function Θ_Λ.



The localization function $\Theta_\Lambda(y) = \frac{1}{1+(\Lambda^{-s}y)^{2n}}$, $0 < s < 1/3$ restricts the condensate to the region **around the origin**.

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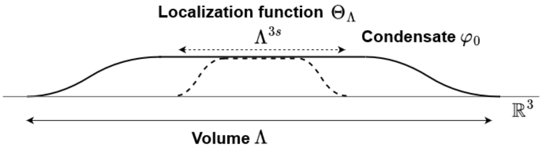

The localization function is $\Theta_\Lambda(y) := \frac{1}{1+(\Lambda^{-s}y)^{2n}}$. $\tilde{\varphi}_t := e^{-it(V*|\varphi_0|^2-\mu_0)}\varphi_0$ conserves the flatness around the origin $|\tilde{\varphi}_t|^2 = |\varphi_0|^2 \sim 1$ [Deckert-Fröhlich-Pickl-Pizzo 16].

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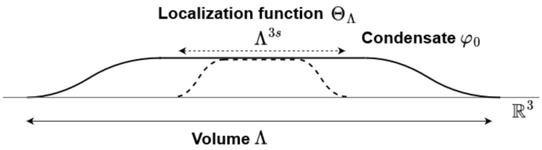

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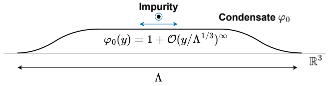
We show that the **condensate remains flat around the origin**, namely $\forall T, \gamma > 0$ and $0 < s < 1/3$ $\exists n, C > 0$ such that for all densities $\rho \geq 1$ and volumes $\Lambda \geq 1$

$$\sup_{t\in[-T,T]} \|\Theta_\Lambda(\varphi_t - \tilde{\varphi}_t)\|_2 \leq C\Lambda^{-\gamma}.$$

Impurity Localization

We show that the **impurity remains** in the **bulk** of the condensate:
$\forall T \geq 0$, $M \in \mathbb{N}_0$ $\exists C_M > 0$ such that for all densities $\rho \geq 1$, volumes $\Lambda \geq 1$

$$\sup_{t\in[-T,T]} \Big\| \underbrace{|x|^{2M}}_{\text{Impurity position}} \cdot \underbrace{\psi_t^{\mathrm{BF}}}_{\text{Polaron dynamics}} \Big\|_{\mathcal{F}^{\mathrm{ex}}} \leq C_M \,.$$



Initial conditions

Control **impurity position** x

$\downarrow$ needs

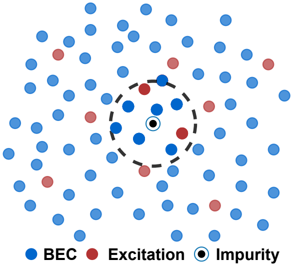
Control **kinetic energy** ∇_x

$\downarrow$

Control energy gained by excitation

$\downarrow$

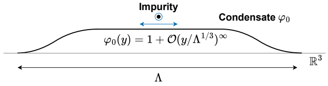
Control **number of excitations** effectively interacting with impurity



Impurity Localization

We show that the **impurity remains** in the **bulk** of the condensate:
$\forall T \geq 0$, $M \in \mathbb{N}_0$ $\exists C_M > 0$ such that for all densities $\rho \geq 1$, volumes $\Lambda \geq 1$

$$\sup_{t\in[-T,T]} \Big\| \underbrace{|x|^{2M}}_{\text{Impurity position}} \cdot \underbrace{\psi_t^{\mathrm{BF}}}_{\text{Polaron dynamics}} \Big\|_{\mathcal{F}^{\mathrm{ex}}} \leq C_M \,.$$



Initial conditions

Control **impurity position** x

$\downarrow$ needs

Control **kinetic energy** ∇_x

$\downarrow$

Control energy gained by excitation

$\downarrow$

Control **number of excitations** effectively interacting with impurity

Initial condition: For all $M \geq 0$

$$\langle \psi_0^{\mathrm{BF}}, (\underbrace{-\Delta_x}_{\text{Kin. energy}} + \underbrace{|x|^{2M}}_{\text{Position}} + \underbrace{U_{\mathcal{V}}(\mathcal{N}_+ + 1)U_{\mathcal{V}}^*}_{\text{Eff. number ex.}})^M \psi_0^{\mathrm{BF}} \rangle \leq C_M.$$

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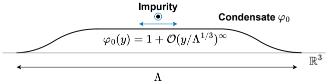

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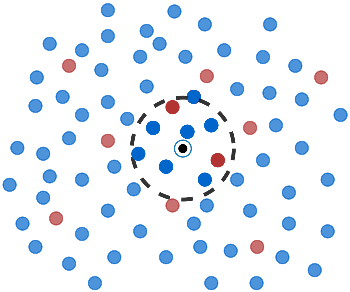
Control of the Local Excitation Number

Reminder: $H^{\mathrm{BF}} = {\color{red}H^{\mathrm{Bog}}} - \frac{\Delta_x}{2m} + a(Q_t W_x \varphi_t) + a^*(Q_t W_x \varphi_t)$.



H^{Bog} creates excitation in the whole volume but the impurity see only the local excitations surrounding it $\rightarrow$ **extract H^{Bog}**.

With the propagator U_t^{Bog} of H^{Bog} ($\partial_t U_t^{\mathrm{Bog}} = \mathrm{i} H^{\mathrm{Bog}} U_t^{\mathrm{Bog}}$) we get

$$\begin{aligned}\widetilde{H}^{\mathrm{BF}} &= U_t^{\mathrm{Bog}} H^{\mathrm{BF}} (U_t^{\mathrm{Bog}})^* + i(\partial_t U_t^{\mathrm{Bog}})(U_t^{\mathrm{Bog}})^* \\ &= {\color{red}0} - \frac{\Delta_x}{2m} + a((\tilde{U}_t - V_t) Q_t W_x \varphi_t) \\ &\quad + a^*((\tilde{U}_t - V_t) Q_t W_x \varphi_t)\,.\end{aligned}$$

For $\widetilde{H}^{\mathrm{BF}}$ we are able to prove

$$\sup_{t \in [-T,T]} \| \underbrace{|x|^{2M}}_{\text{Impurity position}} \cdot \underbrace{\widetilde{\psi}_t^{\mathrm{BF}}}_{= I \otimes U_t^{\mathrm{Bog}} \psi_t^{\mathrm{BF}}} \|_{\mathcal{F}^{\mathrm{ex}}} \le C_M$$

but $|x|^{2M} \otimes I_{\mathcal{F}}$ and $I_{L_x^2} \otimes U_t^{\mathrm{Bog}}$ commute, which proves tracer localization.

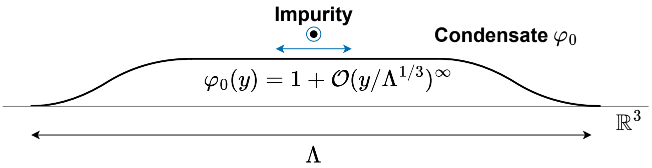
Methods Overview

$$\underbrace{H^{\mathrm{ex}}}_{\text{Full excitation dynamics}} = \underbrace{H^{\mathrm{BF}}}_{\text{Polaron dynamics}} + 1/\sqrt{\rho} \cdot \text{error} + \underbrace{\sqrt{\rho} W * |\varphi_t|^2(x)}_{\text{Mean-field impurity-condensate interaction}} .$$

We want to show

$$H^{\mathrm{ex}} \xrightarrow{\Lambda, \rho \to \infty} H^{\mathrm{BF}} .$$

- Control **excitation number** similar to [Petrat-Pickl-Soffer 20, Lewin-Nam-Schlein 15] $\to \left(1/\sqrt{\rho} \cdot \text{error}\right)$ small.
- ✔ Control **impurity localization** and φ_t remains **flat around the origin** $\to \sqrt{\rho} W * |\varphi_t|^2(x) \sim \sqrt{\rho} W * 1$.



Initial conditions

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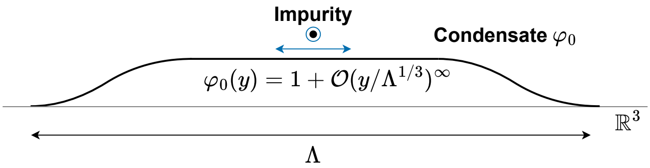

Initial conditions

Conclusion

- **Bose Polaron**: Quasi-particle of impurity and bosons.
- We proved the **effective description** of the **full microscopic dynamics** through the **Polaron dynamics**: $H^{\mathrm{BF}} = H^{\mathrm{Bog}} - \frac{\Delta_x}{2m} + a(Q_t W_x \varphi_t) + a^*(Q_t W_x \varphi_t)$.
- **Impurity** is **localized** in the large Bose gas.

- (Ongoing) Derive effective limiting **dynamics** independent of Λ at **infinite volume**.

Thank you for your attention!

Generalized Initial State

Goal: Initial state with

- $\mathcal{O}(1)$ excitations locally (in unit volume)
- $\mathcal{O}(\Lambda)$ excitations globally (in volume Λ)

Currently: ψ_0^{BF} with $\mathcal{O}(1)$ excitations globally $\to$ $\mathcal{O}(1)$ excitations locally. Needed for tracer localization!
$(\langle \psi_0^{\text{BF}} (-\Delta_x + x^2 + \mathcal{N} + 1)^M \psi_0^{\text{BF}} \rangle \leq C)$
Solution: Assume Tracer localization conditions on the Bogoliubov transformed initial state

$$U_{\mathcal{V}} \psi_0^{\text{BF}}$$

with

$$\underbrace{\|\mathcal{V}\|_{\text{op}} \leq C}_{\mathcal{O}(1) \text{ local excitations}} \quad , \quad \underbrace{\|\mathcal{V}\mathcal{V}^* - 1\|_{HS} \leq C \cdot \Lambda}_{\mathcal{O}(\Lambda) \text{ global excitations}} .$$

Same estimates as before!
$\langle \psi_t^{\text{BF}}, (-\Delta_x + x^2 + U_{\mathcal{V}}(\mathcal{N} + 1) U_{\mathcal{V}}^*)^M \psi_t^{\text{BF}} \rangle \leq C$ if $\psi_{t=0}^{\text{BF}} = U_{\mathcal{V}} \psi_0^{\text{BF}}$ and
$\langle \psi_0^{\text{BF}}, (-\Delta_x + x^2 + U_{\mathcal{V}}(\mathcal{N} + 1) U_{\mathcal{V}}^*)^M \psi_0^{\text{BF}} \rangle \leq C$.

Scaling

Density $\rho \geq 1$ and $\Lambda = \rho^{\alpha}$, the gas is for all $\alpha > 0$ dense!

- The case $\alpha = 0$ is the standard mean-field scaling with fixed volume (e.g., $\Lambda = 1$).
- The case $\alpha = \infty$ corresponds to the thermodynamic limit with constant density (e.g. $\rho = 1$) as the volume grows.
- Our approximation is valid for $0 < \alpha < 1/3$.

Comparison with the β-scaling:

$$-\sum_i \frac{\Delta_{y_i}}{N^{2\beta}} + N^{3\beta-1} \sum_{i,j} V\big(N^{\beta}(y_i - y_j)\big)\,, \quad y_i \in [-1/2, 1/2]^3\,.$$

- $\beta := \frac{\alpha}{3(1+\alpha)}$, $0 < \beta < 1/4$.

Global Excitation Number Control

Karlsruhe Institute of Technology

$$
\begin{aligned}
H^{\mathrm{BF}} &= H^{\mathrm{Bog}} - \frac{\Delta_x}{2m} + a(Q_t W_x \varphi_t) + a^*(Q_t W_x \varphi_t)\,, \\
i\partial_t \psi_t^{\mathrm{BF}} &= H^{\mathrm{BF}} \psi_t^{\mathrm{BF}}\,, \\
H^{\mathrm{Bog}} &\sim d\Gamma(\xi)\,.
\end{aligned}
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For H^{BF} to be valid we need the **propagation** of the following setting from our initial conditions:

Most particle in condensate and **few excitations**.

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Show

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\left\langle \psi_t^{\mathrm{BF}}, \underbrace{(\mathcal{N}_+ + 1)}_{\#\text{ excitations}}{}^n \psi_t^{\mathrm{BF}} \right\rangle \leq C_n \Lambda^n \quad \text{if} \quad \left\langle \psi_0^{\mathrm{BF}}, (\mathcal{N}_+ + 1)^n \psi_0^{\mathrm{BF}} \right\rangle \leq C_n \Lambda^n\,.
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Achieved with methods from [Petrat-Pickl-Soffer 20] and [Lewin-Nam-Schlein 15].

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Show for the global excitation number

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Interpretation

$$\frac{\#\text{excitations}}{\text{volume}} \sim \mathcal{O}(1)\,, \quad \frac{\#\text{particles}}{\text{volume}} \sim \mathcal{O}(\rho) \to \infty\,.$$

Hence a few excitations.