Derivation of the Chern-Simons-Schrödinger equation from the dynamics of an almost-bosonic-anyon gas

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General idea

N-body bosonic system with Hamiltonian H_N on $L^2_{sym}(\mathbb{R}^{dN})$,

- Micro $\rightarrow$ meso/macro description, $u \in L^2(\mathbb{R}^d)$

$$\text{Ground State :} \quad \inf_{\substack{\Psi_N \in L^2_{sym}(\mathbb{R}^{dN}),\\ \|\Psi_N\|=1}} \langle \Psi_N | H_N \Psi_N \rangle \rightarrow N \inf_{\substack{u \in L^2(\mathbb{R}^d),\\ \|u\|=1}} \mathcal{E}[u]$$

$$\text{Dynamics :} \quad \mathrm{i}\partial_t \Psi_N(t) = H_N \Psi_N(t) \rightarrow \mathrm{i}\partial_t u(t) = \mathcal{H} u(t)$$

- Bose-Einstein condensate $\Psi_N \simeq \otimes_{j=1}^N u_j \simeq$ GS :

$$H_N \Psi_N = E \Psi_N \quad \rightarrow \quad \Psi_N(t) = e^{-\mathrm{i}Et} \Psi_N(0)$$

- Heisenberg motion for $\Gamma_N(t) = |\Psi_N(t)\rangle \langle \Psi_N(t)|$

$$\mathrm{i}\partial_t \Gamma_N(t) = [H_N, \Gamma_N(t)]$$

- Reduced density matices $\gamma_N^{(k)} = \mathrm{Tr}_{k+1 \rightarrow N} [\Gamma_N] \simeq |u\rangle \langle u|^{\otimes k}$

The Anyon Gas

Leinaas and Myrheim (1997), anyons are 2D particles s.t :

$$\tilde{\Psi}(x_1, ..., x_j, ..., x_k, ..., x_N) = e^{\mathrm{i}\alpha\pi}\tilde{\Psi}(x_1, ..., x_k, ..., x_j, ..., x_N)$$

Magnetic gauge picture with Ψ a bosonic wave function.

$$H_{N,R} = \sum_{j=1}^{N}(-\mathrm{i}\nabla_j + \alpha \mathbf{A}^R(x_j))^2. \tag{1}$$

Hamiltonian for N extended anyons of radius R where

$$\mathbf{A}(x_1) = \sum_{k\neq 1} \frac{(x_1 - x_k)^{\perp}}{|x_1 - x_k|^2}, \quad \mathbf{B}(x_1) = \boldsymbol{\nabla} \times \mathbf{A} = 2\pi \sum_{k \neq j} \delta(x_1 - x_k) \tag{2}$$

Anyons carry an Aharonov-Bohm magnetic flux of strength α.
Almost-bosonic limit: $\alpha = \beta(N-1)^{-1}$ and $H_{N,R}$ acts on $L^2_{\mathrm{sym}}(\mathbb{R}^{2N})$.

Average-field limit

Hamiltonian for N trapped almost-bosonic extended anyons:

$$
\begin{aligned}
H_{R,N} = &\sum_{j=1}^{N} (-\mathrm{i}\nabla_j)^2 + V(x_j) \text{ “Kinetic and potential terms"} \\
&+ \alpha \sum_{j\neq k} \left(-\mathrm{i}\nabla_j . \nabla^\perp w_R(x_j - x_k) + \text{h.c}\right) \text{ “Mixed"} \\
&+ \alpha^2 \sum_{j\neq k\neq \ell} \nabla^\perp w_R(x_j - x_k) . \nabla^\perp w_R(x_j - x_\ell) \text{ “Three-body"} \\
&+ \alpha^2 \sum_{j\neq k} |\nabla^\perp w_R(x_j - x_k)|^2 \text{ “Singular two-body".}
\end{aligned}
$$

The average-field energy, Lundholm, Rougerie (2016), G (2021)

$$
\inf \frac{H_{N,R}}{N} \to \inf \mathcal{E}_R^{\mathrm{af}}[u] := \inf \int_{\mathbb{R}^2} |(-\mathrm{i}\nabla + \beta \mathbf{A}^R[|u|^2])u|^2 + V|u|^2 \tag{3}
$$

where $\mathbf{A}^R[|u|^2] := \nabla^\perp w_R * |u|^2$ and $R > 1/\sqrt{N}$.

The effective equation

- What equation should drive $u(x,t)$?
- BBGKY with $\gamma_N^{(k)}(t) = |u(t)\rangle \langle u(t)|^{\otimes k}$.

Definition (Chern–Simons–Schrödinger equation)

We denote by $\mathrm{CSS}(R, u_0)$ the differential problem whose the unknown is $u : (\mathbb{R}_+ \times \mathbb{R}^2) \mapsto \mathbb{C}$ satisfying

$$\begin{aligned} \mathrm{i}\partial_t u &= \left(-\mathrm{i}\nabla + \beta \mathbf{A}^R\left[|u|^2\right]\right)^2 u \\ &\quad - \beta\left[\nabla^\perp w_R * \left(2\beta \mathbf{A}^R\left[|u|^2\right]|u|^2 + \mathrm{i}\left(u\nabla\overline{u} - \overline{u}\nabla u\right)\right)\right] u \end{aligned} \tag{4}$$

with initial condition $u(0,x) = u_0(x) \in H^2(\mathbb{R}^2)$.

- There exists a $T > 0$ such that $u(t) \in C([0,T], H^2(\mathbb{R}^2))$.

The result

Theorem (Convergence to CSS, J.Lee, G (2024))

Let φ_t be the solution of $\mathrm{CSS}(\varphi_0)$ *with initial data* $\varphi_0 \in H^2(\mathbb{R}^2)$*. Take* $R = (\log N)^{-\frac{1}{2}+\varepsilon}$ *and* $\Psi_N(0) = \varphi_0^{\otimes N}$*. Then, there exist constants* $T, C, c > 0$ *depending on* $\|\varphi_0\|_{H^2}$ *such that for any* $|\beta| \leq c$ *and for all time* $0 \leq t \leq T$*, we have, for sufficiently large* N*,*

$$\mathrm{Tr}\left|\gamma_N^{(k)}(t) - \left|\varphi_t^{\otimes k}\right\rangle\left\langle\varphi_t^{\otimes k}\right|\right| \leq C(\log N)^{-\frac{1}{2}+\varepsilon} \tag{5}$$

$\forall k$ *and for any choice of* $\varepsilon > 0$*.*

- Could be global in time but $\left\|\varphi_t^R\right\|_{H^2} \leq R^{-Ct} \to |\beta| \leq c$
- If $g > 0$, CSS is globally well-posed in H^s, $s \geq 1$
- One and a half step to get a polynomial rate $R \simeq N^{-\delta}$, $\delta > 0$
- Eq. (5) valid with polynomial decay if R is kept fix

Outline of the proof

Let $\varphi(t)$ be solution of $\mathrm{CSS}(R, \varphi_0)$ and define

$$p_1(t) := |\varphi_t(x_1)\rangle\langle\varphi_t(x_1)| \quad \text{and} \quad q_1(t) := \mathbb{1}_{L^2(\mathbb{R}^2)} - p_1(t). \tag{6}$$

We have to control

$$\mathcal{N}_+(t) := \langle \Psi_N | q_1(t) \Psi_N \rangle \tag{7}$$

We can compute

$$\partial_t \mathcal{N}_+(t) = -\mathrm{i} \left\langle \Psi_N(t) \middle| \left[H_{N,R} - \sum_{j=1}^{N} \mathcal{H}(x_j), q_1 \right] \Psi_N(t) \right\rangle \tag{8}$$

where the Hartree Hamiltonian is

$$\begin{aligned}\mathcal{H}(x) :=& \left(-\mathrm{i}\nabla + \beta \mathbf{A}^R \left[|\varphi_t|^2\right]\right)^2 (x) \\ & - \beta \left[\nabla^{\perp} w_R * \left(2\beta \mathbf{A}^R \left[|\varphi_t|^2\right] |\varphi_t|^2 + \mathrm{i}\left(\varphi_t \nabla \overline{\varphi_t} - \overline{\varphi_t} \nabla \varphi_t\right)\right)\right](x).\end{aligned}$$

The mean-field cancellation

- We split the space as

$$\mathbb{1}_{L^2(\mathbb{R}^4)} = (p_1 + q_1)(p_2 + q_2) \tag{9}$$

$$\begin{aligned} \partial_t \mathcal{N}_+(t) = -\mathrm{i} \left\langle p_1 p_2 \middle| \cdots , q_1 p_2 \right\rangle_{\Psi_N} - \mathrm{i} \left\langle p_1 p_2 \middle| \cdots , q_1 q_2 \right\rangle_{\Psi_N} \\ - \mathrm{i} \left\langle q_1 p_2 \middle| \cdots , q_1 q_2 \right\rangle_{\Psi_N} \end{aligned}$$

$$p_2 p_3 H_{N,R} p_3 p_2 - \mathcal{H}(x_1) p_2 p_3 \simeq \mathit{small} \tag{10}$$

- For instance, we have

$$p_2 \nabla_1 \cdot \nabla^{\perp} w_R(x_1 - x_2) p_2 = \nabla_1 \cdot \langle \varphi(t) | w_R(x_1 - \cdot) \varphi(t) \rangle \, p_2 = \nabla_1 \cdot \mathbf{A}^R[\rho] p_2$$

- In the others, we have many $q \simeq \mathcal{N}_+$ to close the Grönwall.

Control of the kinetic energy

- How to bound $\left\langle q_1 p_2 \middle| \nabla_1 \cdot \mathbf{A}^R[\rho], q_1 q_2 \right\rangle_{\Psi_N}$?

- We need to control $q_1 \Delta_1 q_1$:

$$\|\nabla_1 q_1 \Psi_N\|^2 \leq \sqrt{\mathcal{N}_+} + \frac{1}{N} \left| \langle \Psi_M(t) | H_{R,N} \Psi_N(t) \rangle - \mathcal{E}_R^{\mathrm{af}}[\varphi(t)] \right| \quad (11)$$

- If we do a Grönwall on $\sqrt{\mathcal{N}_+}$,

$$\mathrm{i}\partial_t \sqrt{\mathcal{N}_+} \leq C_{\|\varphi(t)\|_{H^2}} \sqrt{\mathcal{N}_+} + \|\nabla_1 q_1 \Psi_N\|^2 + \frac{1}{N}.$$

- Control on our operators $\nabla^\perp w_R \simeq |x|^{-1}$, via

$$(\nabla_1 \cdot \nabla^\perp w_R(x_1 - x_2) + \mathrm{h.c})^2 \leq |\log R|^2 (\mathbb{1} - \Delta_1)(\mathbb{1} - \Delta_2)$$
$$\nabla^\perp w_R(x_1 - x_2) \cdot \nabla^\perp w_R(x_1 - x_3) \leq (\mathbb{1} - \Delta_1)$$

Technical realization of $\mathcal{N}_+^p$

- Split the space $\mathbb{1}_{L^2(\mathbb{R}^{2N})} = (p_1+q_1)(p_2+q_2)\cdots(p_N+q_N)$,
- $P_N^{(k)}$ collects all summands containing k factors of q operators.

$$P_N^{(k)} := \sum_{\substack{a\in\{0,1\}^N \\ \sum_j a_j = k}} \prod_{j=1}^{N} p_j^{1-a_j} q_j^{a_j}. \tag{12}$$

- $\sum_{k=0}^{+\infty} P_N^{(k)} = \mathbb{1}_{L^2(\mathbb{R}^{2N})}$, $P_N^{(k)} P_N^{(l)} = \delta_{kl} P_N^{(k)}$, $P_N^{(0)} = p^{\otimes N}$.
- Define

$$\hat{m}(\xi) := \sum_{k=1}^{N} \left(\frac{k}{N}\right)^{\xi} P_N^{(k)}, \quad \hat{m}(1) = \mathcal{N}_+, \quad \hat{m}(1/2) = \sqrt{\mathcal{N}_+}. \tag{13}$$

- Properties $\hat{m}(1/2)\hat{m}(1/2) = \hat{m}(1)$, $\hat{m}^{(n)}(1/2) \leq \frac{n}{N}\hat{m}(-1/2)$.

Thank you for your attention