Jacobs University
Fall 2018

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Advanced Calculus

Some extra exercises for part II with solutions

Note: Here I sometimes just provide the final results. In exams you have to provide detailed steps and explanations for your solution.

Problem 1 (Taylor Series)
Compute the Taylor series around $x = 0$ for $|x| \leq 1$ of

$$f(x) = \arctan(x).$$

Solution:

$$\arctan(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}.$$

Problem 2 (Maxima/Minima)
Find all maxima, minima, and points of inflection of

$$f(x) = x^3 - 3x + 3$$

and determine where the function is concave and where it is convex. Based on your results, qualitatively sketch the graph of f.

Solution: Maximum at -1, minimum at $+1$, point of inflection at 0. For $x < 0$ the function is concave, for $x > 0$ it is convex.

Problem 3 (Integration by Substitution)
Compute for any $a < b$

$$\int_a^b \Big((x-a)(b-x)\Big)^{-1/2} dx \quad \text{and} \quad \int_a^b \Big((x-a)(b-x)\Big)^{1/2} dx$$

using the substitution $x = a\cos^2 y + b \sin^2 y$.

Solution:

$$\int_a^b \Big((x-a)(b-x)\Big)^{-1/2} dx = \pi,$$

$$\int_a^b \Big((x-a)(b-x)\Big)^{1/2} dx = \frac{\pi(b-a)^2}{8}.$$

Problem 4 (Integration by Parts)
Compute the integrals

$$\int x^2 e^{2x}\,dx, \qquad \int (\ln(x))^2\,dx.$$

Solution:

$$\int x^2 e^{2x}\,dx = \frac{e^{2x}(2x^2 - 2x + 1)}{4} + C,$$

$$\int (\ln(x))^2\,dx = x\ln(x)^2 - 2x\ln(x) + 2x + C.$$

Problem 5 (Improper Integrals)
Compute the following improper integrals, in case they exist.

$$\int_0^1 \frac{1}{x^2}\,dx, \qquad \int_1^\infty \frac{1}{x^2}\,dx, \qquad \int_0^1 \frac{x}{(1-x^2)^{1/2}}\,dx.$$

Solution:

$$\int_0^1 \frac{1}{x^2}\,dx \quad \text{does not exist,}$$

$$\int_1^\infty \frac{1}{x^2}\,dx = 1,$$

$$\int_0^1 \frac{x}{(1-x^2)^{1/2}}\,dx = 1.$$

Problem 6 (Uniform Convergence and Exchange of Limits)
Consider the sequence of functions

$$f_n(x) = \begin{cases} n & \text{, for } 0 < x \leq \frac{1}{n} \\ 0 & \text{, otherwise.} \end{cases}$$

What is the pointwise limit $n \to \infty$? What is $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx$? What is $\int_0^1 \lim_{n\to\infty} f_n(x)\,dx$? Does f_n converge uniformly to some function f?

Solution: $f_n(x)$ converges to 0 for each fixed x. Furthermore, $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 1$, but $\int_0^1 \lim_{n\to\infty} f_n(x)\,dx = 0$. But, according to the theorem from class, if f_n converges to $f = 0$ uniformly, then both integrals give the same result. So f_n does not converge uniformly to $f = 0$ (and of course not to any other f).

Problem 7 (ODEs: Separation of Variables)
Solve the ODE
$$\frac{dy}{dx} = x + xy$$
by separation of variables.

Solution:
$$y(x) = Ce^{x^2/2} - 1.$$

Problem 8 (Linear ODEs)
Give the general solution to the linear homogeneous ODE
$$y'' + y' - 2y = 0.$$
Then give the solution for the initial condition $y(0) = 2$ and $y'(0) = 5$. What is the behavior of the solution as $x \to \infty$? Also find one particular solution to the linear inhomogeneous ODE
$$y'' + y' - 2y = e^{-x}.$$
Finally, provide the general solution (i.e., involving two constants) to this inhomogeneous ODE.

Solution: The general solution to the homogeneous ODE is $y(x) = ae^x + be^{-2x}$, where a, b are two constants. With the above initial conditions we find $a = 3$, $b = -1$, so the solution for the given initial conditions is $y(x) = 3e^x - e^{-2x}$. This solution diverges to $+\infty$ as $x \to \infty$.

One solution to the inhomogeneous ODE can be found by using Ae^{-x} as ansatz. Then we find that $y_{\text{part}}(x) = -\frac{1}{2}e^{-x}$ is one particular solution (it does not involve any constants). So the general solution to the inhomogeneous ODE is
$$y_{\text{gen}}(x) = ae^x + be^{-2x} - \frac{1}{2}e^{-x}.$$

Problem 9 (Fourier Series)
Consider the 2π-periodic function f which is $f(x) = \cosh(x - \pi)$ on the interval $[0, 2\pi]$. Does its Fourier series converge uniformly to f? Compute the Fourier series of f. Then, by evaluating f and its Fourier series at π, compute the value of the series
$$\sum_{k=1}^{\infty} \frac{(-1)^k}{1 + k^2}.$$

Solution: The function is continuous and piecewise continuously differentiable, so the Fourier series converges to f uniformly. The Fourier coefficients are
$$c_k = \frac{1}{2\pi} \int_0^{2\pi} \cosh(x - \pi)e^{-ikx}\, dx = \frac{\sinh(\pi)}{\pi(1 + k^2)},$$

for all k (also $k = 0$). Then the Fourier series is

$$\mathcal{F}[f](x) = \sum_{k=-\infty}^{\infty} \frac{\sinh(\pi)}{\pi(1+k^2)} e^{ikx}.$$

Since we have uniform convergence, we have that $\mathcal{F}[f](\pi) = f(\pi)$. We find

$$1 = f(\pi) = \cosh(\pi - \pi) = \mathcal{F}[f](\pi) = \sum_{k=-\infty}^{\infty} \frac{\sinh(\pi)}{\pi(1+k^2)} (-1)^k.$$

Solving for the desired sum, we get

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{1+k^2} = \frac{\pi}{2\sinh(\pi)} - \frac{1}{2}.$$