Jacobs University
Fall 2018

October 17, 2018

Advanced Calculus

Some extra exercises for part I with solutions

Note: Here I sometimes just provide the final results. For your homeworks and exams you have to provide detailed steps and explanations for your solution.

Problem 1 (Binomial Coefficients)

Compute

$$\sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

for any n and $0 < p < 1$.

Solution:

$$\sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} = np$$

as discussed in class. This is the expectation value of the binomial distribution (e.g., it tells you how many heads you get on average for n coin tosses, given that the probability for heads in one coin toss in p).

Problem 2 (Induction)

Prove by induction that

$$\sum_{k=0}^{n} k^3 = \frac{1}{4} n^2 (n+1)^2.$$

Solution: Direct proof using induction.

Problem 3 (Polynomials)

Factorize the polynomial $p(x) = x^3 - 3x^2 - 13x + 15$.

Solution:

$$p(x) = (x-1)(x+3)(x-5).$$

The $(x-1)$ factor can be guessed; then figure out a and b in

$$x^3 - 3x^2 - 13x + 15 = (x-1)(x^2 + ax + b)$$

and find the roots of $x^2 + ax + b$.

Problem 4 (Sequences and Convergence)
Show and carefully explain why the sequence

$$a_n = \frac{4n^3 + 3n}{(\sqrt{n+1} - \sqrt{n})n^{7/2}}$$

converges, and what its limit is.

Solution:

$$\begin{aligned} a_n &= \frac{4n^3 + 3n}{(\sqrt{n+1} - \sqrt{n})n^{7/2}} = \frac{(4n^3 + 3n)(\sqrt{n+1} + \sqrt{n})}{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})n^{7/2}} \\ &= \frac{(4n^3 + 3n)(\sqrt{n+1} + \sqrt{n})}{n^{7/2}} \\ &= \underbrace{(4 + 3n^{-2})}_{\to 4}\underbrace{(\sqrt{1 + n^{-1}} + 1)}_{\to 2} \\ &\to 8. \end{aligned}$$

Problem 5 (Sequences and Convergence)
Determine $\liminf_{n\to\infty} a_n$ and $\limsup_{n\to\infty} a_n$ of the sequence

$$a_n = (-2)^n \left(2^{-n+1} + 10^{-n}\right).$$

Does $\lim_{n\to\infty} a_n$ exist?

Solution:

$$\liminf_{n\to\infty} a_n = -2, \qquad \limsup_{n\to\infty} a_n = 2,$$

and since those two are different the limit $n \to \infty$ does not exist.

Problem 6 (Infinite Series)
Compute

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k+3)}$$

or show that the limit does not exist.

Solution:

$$\frac{1}{(2k-1)(2k+3)} = \frac{1}{4}\left(\frac{1}{2k-1} + \frac{1}{2k+3}\right).$$

So if we define $a_n = \frac{1}{2k-1}$ we have

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \frac{1}{(2k-1)(2k+3)} = \frac{1}{4}\sum_{k=1}^{n}(a_k - a_{k+2}) = \frac{1}{4}(a_1 + a_2 - a_n - a_{n+2}) \\ &\to \frac{1}{4}(a_1 + a_2) = \frac{1}{4}\left(1 + \frac{1}{3}\right) = \frac{1}{3}. \end{aligned}$$

Problem 7 (Power Series)
Determine the radius of convergence ρ for the power series

$$P(x) = \sum_{k=1}^{\infty} \frac{1}{k^2} x^k$$

and state whether it converges at $x = \pm\rho$ or not. What is the derivative $P'(x)$? Does it converge at $x = \pm\rho$ or not?

Solution: By root or ratio test, the radius of convergence is $\rho = 1$. Then

$$P(1) = \sum_{k=1}^{\infty} \frac{1}{k^2} \quad \text{converges absolutely,}$$

$$P(-1) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \quad \text{converges absolutely,}$$

e.g., because as we showed in a previous homework, $\sum_{k=2}^{\infty} \frac{1}{k(k-1)}$ converges; then we can use the comparison test appropriately (later when we have the integral test we will see directly that the series converges). Also,

$$P'(x) = \sum_{k=1}^{\infty} \frac{1}{k} x^{k-1},$$

and

$$P'(1) = \sum_{k=1}^{\infty} \frac{1}{k} \quad \text{diverges (comparison test),}$$

$$P'(-1) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \quad \text{converges conditionally (Leibniz test).}$$

Problem 8 (Complex Numbers)
Find all roots of the equation

$$z^3 + 2 = 0.$$

Solution:

$$z = (-2)^{1/3} = (2e^{i\pi + 2\pi ik}) = \sqrt[3]{2}e^{i\pi/3 + 2\pi ik/3},$$

so the three roots are

$$z_1 = \sqrt[3]{2}e^{i\pi/3}, \quad z_2 = \sqrt[3]{2}e^{i\pi}, \quad z_3 = \sqrt[3]{2}e^{i5\pi/3}.$$

Problem 9 (Complex Numbers)
Carefully derive the trigonometric identity

$$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

using Euler's formula.

Solution: Rewrite

$$e^{i(x+y)} = e^{ix}e^{iy}$$

using Euler's formula, and then take the imaginary part on both sides.

Problem 10 (Derivatives)
Consider the function

$$f(x) = \frac{\ln(x)}{x - 3}.$$

What are the domain, image and derivative of f?

Solution: The domain is $(0, \infty) \setminus \{3\}$ and the image is $\mathbb{R}$. Also,

$$f'(x) = \frac{x^{-1}(x - 3) - \ln(x)}{(x - 3)^2}.$$

Problem 11 (Derivatives)
Compute the derivatives of

$$f(x) = \sin(x)\cos(x), \quad \text{and} \quad g(x) = \arcsin(x).$$

Solution:

$$f'(x) = \cos^2(x) - \sin^2(x), \qquad g'(x) = \frac{1}{\sqrt{1 - x^2}}.$$