Jacobs University
Fall 2018

November 5, 2018

Advanced Calculus

Homework 7

Due on November 12, 2018

Problem 1 [6 points]

Let $f : [1, \infty) \to [0, \infty)$ be nonincreasing (i.e., for $x \le y$ we have $f(x) \ge f(y)$).

(a) Show that

$$\sum_{k=2}^{n} f(k) \le \int_{1}^{n} f(x)\,dx \le \sum_{k=1}^{n} f(k).$$

Here you can use the fact that nonincreasing functions are integrable.

(b) Show that $\sum_{k=1}^{n} \frac{1}{k}$ diverges logarithmically for large n. (Note: The constant $\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln(n) \right)$ is called Euler-Mascheroni constant.)

(c) Show that $\sum_{k=1}^{\infty} \frac{1}{k^a}$ converges for all $a > 1$.

(d) Does $\sum_{k=2}^{\infty} \frac{1}{k \ln(k)}$ converge or diverge? What about $\sum_{k=2}^{\infty} \frac{1}{k(\ln(k))^b}$ for $b > 1$? (*Hint: substitution.*)

Problem 2 [6 points]

Compute the integrals

(a)

$$\int \frac{\cos(\ln(x))}{x}\,dx \quad \text{for } x > 0,$$

(b)

$$\int x^2 \sin(x)\,dx,$$

(c)

$$\int \frac{1}{x^2 + 2x + 6}\,dx,$$

(d)

$$\int \frac{\sin(2x)}{1 + 4\sin^2(x)}\,dx.$$

Problem 3 [5 points]
In class we encountered the integral

$$L = \frac{1}{2}\int_0^2 \sqrt{1+y^2}\,dy.$$

(a) Writing $\sqrt{1+y^2}$ as $1\sqrt{1+y^2}$, use integration by parts in order to express $\int \sqrt{1+y^2}\,dy$ in terms of $\int (1+y^2)^{-1/2}\,dy$ and some other function.

(b) Then compute

$$\int (1+y^2)^{-1/2}\,dy$$

by using the substitution $y = \sinh(x)$ (see Homework 3).

(c) Put parts (a) and (b) together to compute L.

Problem 4 [3 points]
Let f and g be integrable functions. Prove the Cauchy-Schwarz inequality

$$\left|\int_a^b f(x)g(x)\,dx\right| \le \sqrt{\int_a^b f(x)^2\,dx}\,\sqrt{\int_a^b g(x)^2\,dx}.$$

Hint: Start from the fact that the integral of $\big(f(x) - \lambda g(x)\big)^2$ is bigger or equal zero for all real λ.