Session 5
Sep. 17, 2018

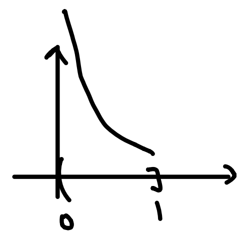
• Maximum Thm.:

If $f:[a,b]\to\mathbb{R}$ is continuous, then it has a maximum and minimum in $[a,b]$.

Note: • domain $=[a,b]$ is important here
• a subset of $\mathbb{R}$ that is closed and bounded is called compact.
(very important concept → later...)

Ex. (of where thm. doesn't hold):

• $f:(0,1]\to\mathbb{R},\ x\mapsto f(x)=\frac{1}{x}$ (domain not closed)

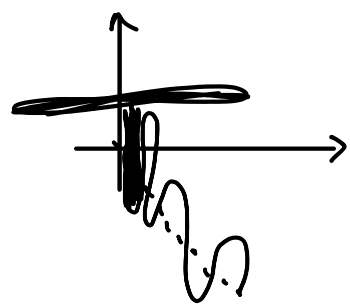


• $f:\mathbb{R}\to\mathbb{R},\ x\mapsto f(x)=x$ (domain not bounded)

• $f:[0,2]\to\mathbb{R},\ x\mapsto f(x)=\begin{cases}\frac{1}{x-1}, & \text{for } x\neq 1\\ 0, & x=1\end{cases}$ (not cont. at $x=1$)

• $f:[0,1]\to\mathbb{R},\ x\mapsto f(x)=\begin{cases}-3x+\sin\left(\frac{1}{x}\right), & x\neq 0\\ 0, & x=0\end{cases}$ (bounded, but not cont.)

e.g., $\sin\Big(\underbrace{\frac{\pi}{2}+1000\cdot 2\pi}_{=\frac{1}{x}}\Big)=1 \Rightarrow f(x)<1\ \forall x\in[0,1]$



Limits of functions:

$f: A \to \mathbb{R}$, then $\lim\limits_{x\to a} f(x) = b$ if $\forall$ sequences $(a_n)_{n\in\mathbb{N}}$

with $a_n \in A$, $a_n \neq a$ $\forall n \in \mathbb{N}$, and $\lim\limits_{n\to\infty} a_n = a$, also

(interesting if $a \notin A$)

$f(a_n) \xrightarrow{n\to\infty} b$

one-sided limits:

left-sided

$\lim\limits_{x\to a^+} f(x) = \lim\limits_{x \searrow a} f(x)$ if additionally $a_n > a$ $\forall n \in \mathbb{N}$

$\lim\limits_{x\to a^-} f(x) = \lim\limits_{x \nearrow a} f(x)$ if additionally $a_n < a$ $\forall n \in \mathbb{N}$

right-sided

Ex.: $\cdot$ $f: \mathbb{R}\setminus\{3\} \to \mathbb{R}$, $x \mapsto f(x) = \dfrac{x^2-9}{x-3}$

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{n\to\infty} \frac{a_n^2-9}{a_n-3} = \lim_{n\to\infty} \frac{(a_n-3)(a_n+3)}{a_n-3}$$

$(a_n \neq 3, a_n \to 3)$

$$= \lim_{n\to\infty} (a_n+3)$$

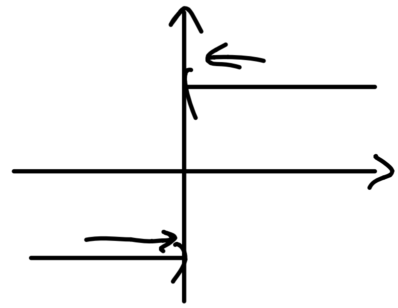
$$= 6$$

$\cdot$ $f(x) = \dfrac{x}{|x|}$, $\lim\limits_{x\to 0^-} \dfrac{x}{|x|} = \lim\limits_{\substack{n\to\infty \\ a_n \to 0 \\ a_n < 0\ \forall n}} \overbrace{\dfrac{a_n}{|a_n|}}^{=-1\ \forall n} = -1$

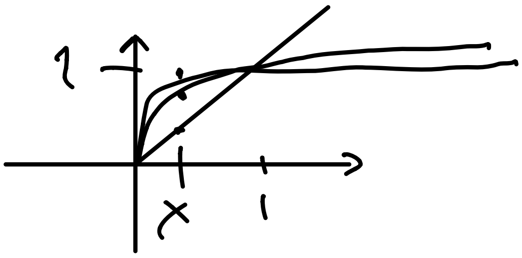
$(f: \mathbb{R}\setminus\{0\} \to \mathbb{R})$

$$\lim_{x \to 0^+} \frac{x}{|x|} = 1$$

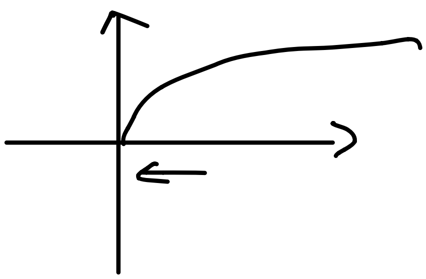
$$\lim_{x \to 0} \frac{x}{|x|} \text{ doesn't exist}$$



- fix $x > 0$, $\lim_{y \searrow 0} x^y = 1$ (proof later)



- fix $y > 0$, $\lim_{x \searrow 0} x^y = 0$ (proof later)



- $f : [-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{0\} \to \mathbb{R}$, $x \mapsto f(x) = \dfrac{3x^2 + 5x}{\sin x}$

later: $\lim_{x \to 0} \dfrac{3x^2 + 5x}{\sin x} = 5$

1.5 Infinite Series

$S_n := \sum_{k=0}^{n} a_k$ $(k=1)$ are called partial sums

<u>Ex.:</u> • geometric series: $\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$

• arithmetic series: $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$

one (more) method to compute series: difference method

suppose $a_k = b_k - b_{k-1}$ then

$$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} (b_k - b_{k-1}) = (b_1 - b_0) + (b_2 - b_1) + (b_3 - b_2) + \ldots + (b_n - b_{n-1})$$

$$= -b_0 + b_n$$

$$= \sum_{k=1}^{n} b_k - \underbrace{\sum_{k=1}^{n} b_{k-1}}_{\sum_{k=0}^{n-1} b_k} = \underbrace{b_n + \sum_{k=1}^{n-1} b_k}_{\sum_{k=1}^{n} b_k} \underbrace{- b_0 - \sum_{k=1}^{n-1} b_k}_{-\sum_{k=0}^{n-1} b_k}$$

<u>Ex.:</u> $\sum_{k=1}^{n} \left(k^3 - (k-1)^3\right) = n^3$

now: let $n \to \infty$ for the sequence $(S_n)_{n \in \mathbb{N}}$

infinite series: $\sum_{k=0}^{\infty} a_k := \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=0}^{n} a_k$

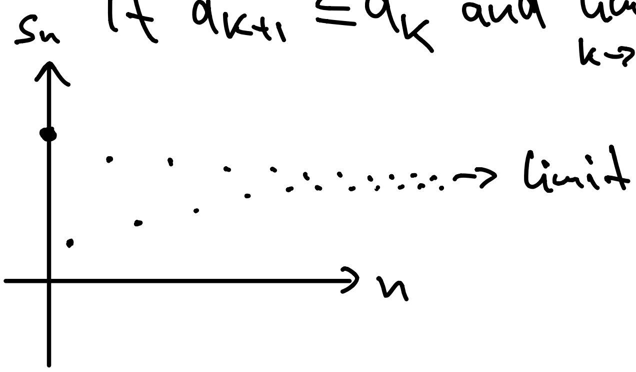
<u>Ex.:</u> $|x|<1$ we have $\sum_{k=0}^{\infty} x^k = \lim_{n\to\infty} \frac{1-x^n}{1-x} = \frac{1}{1-x}$

Does $\sum_{k=0}^{\infty} a_k$ converge? Need convergence criteria.

One criterion:

<u>Leibniz:</u> consider $\sum_{k=0}^{\infty} (-1)^k a_k$ with $a_k > 0 \ \forall\, k \geq 0$.

If $a_{k+1} \leq a_k$ and $\lim_{k\to\infty} a_k = 0$, then the series converges.



Proof: next time