Course evaluation + personal feedback

Midterm: Wed., Oct. 24, 11:15-12:30 LH Res. I

Session 12
Oct. 15, 2018

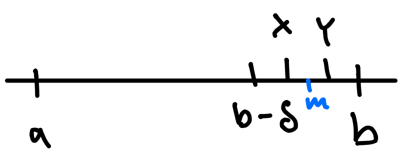
↳ be on time!

↳ no notes, calculators, electronic devices

↳ all material including HW sheet 5

↳ note the university's strict make-up policy (email + doctor's note)

Proof of L'Hospital thm.:



take any (small) $\delta > 0$ and x, y s.t. $a < b-\delta < x < y < b$

then apply Cauchy thm.: $\exists\, m \in (x,y)$ s.t. $\dfrac{f(x)-f(y)}{g(x)-g(y)} = \dfrac{f'(m)}{g'(m)}$

let $y \to b^- \Rightarrow \lim\limits_{y\to b^-} f(y) = 0 = \lim\limits_{y\to b^-} g(y)$

so $\lim\limits_{y\to b^-} \dfrac{f(x)-f(y)}{g(x)-g(y)} = \dfrac{f(x)}{g(x)} = \dfrac{f'(m)}{g'(m)}$

let $x \to b^-$, then also $m \to b^-$ $(x < m < y)$ and taking this limit gives

$$\lim_{x\to b^-} \frac{f(x)}{g(x)} = \lim_{m\to b^-} \frac{f'(m)}{g'(m)}. \qquad \square$$

Examples:

- $\lim\limits_{x\to 0} \frac{\sin x}{x} = \lim\limits_{x\to 0} \frac{\cos x}{1} = 1$

- $\lambda > 0, n \in \mathbb{N}$, $\lim\limits_{x\to\infty} \frac{e^{\lambda x}}{x^n} = \lim\limits_{x\to\infty} \frac{\lambda e^{\lambda x}}{n x^{n-1}} = \ldots = \lim\limits_{x\to\infty} \frac{\lambda^n e^{\lambda x}}{n!} \to \infty$

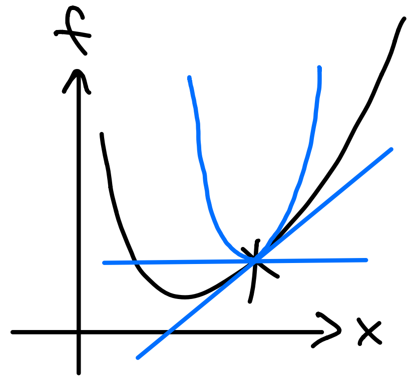
- $\alpha > 0$, ($\alpha \in \mathbb{R}$) $\lim\limits_{x\to\infty} \frac{\ln x}{x^\alpha} = \lim\limits_{x\to\infty} \frac{\frac{1}{x}}{\alpha x^{\alpha-1}} = \lim\limits_{x\to\infty} \frac{1}{\alpha x^\alpha} = 0$

- $\lim\limits_{x\to 0} x \ln x = \lim\limits_{x\to 0} \frac{\ln x}{\frac{1}{x}} = \lim\limits_{x\to 0} \frac{\frac{1}{x}}{-x^{-2}} = \lim\limits_{x\to 0} (-x) = 0$

- $\lim\limits_{x\to 0} x^x = \lim\limits_{x\to 0} e^{x \ln x} = e^0 = 1$

- $\lim\limits_{x\to\infty} x^{\frac{1}{x}} = \lim\limits_{x\to\infty} e^{\frac{1}{x}\ln x} = e^0 = 1$

2.4 Taylor Expansion



expand $f(x)$ in a power series around some a:

first approximation: $f_{T,1}(x) = f(a)$

better: $f_{T,2}(x) = f(a) + (x-a) f'(a)$

↳ here $f_{T,2}(a) = f(a)$ and $f'_{T,2}(x) = f'(a)$

even better: $f_{T,3}(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2} f''(a)$

↳ here $f_{T,3}(a) = f(a)$, $f'_{T,3}(x) = f'(a) + (x-a) f''(a)$

so $f_{T,3}'(a) = f'(a)$, $f_{T,3}''(x) = f''(a)$

(here $f''(x) = f^{(2)}(x) = \frac{d}{dx}\frac{df}{dx}$ is the second derivative)

<u>Theorem (Taylor):</u>

Let f be continuous on $[a,x]$ and $(n+1)$ times differentiable in (a,x). Then

$$\exists m_x \in (a,x) \text{ s.t. } f(x) = \sum_{k=0}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) + \underbrace{\frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(m_x)}_{\text{remainder}}.$$

note: • If f is arbitrarily often differentiable and

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(m_x) \xrightarrow[n\to\infty]{} 0,$$ then we can write f as the

power series $f(x) = \sum_{k=0}^{\infty} \frac{(x-a)^k}{k!} f^{(k)}(a)$, called Taylor series

• If $a=0$, then Taylor series is also called Maclaurin series

<u>Proof:</u> define $R_n(x) = f(x) - \sum_{k=0}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a)$

$$S_n(x) = \frac{(x-a)^{n+1}}{(n+1)!}$$

to show: $R_n(x) = S_n(x)\, f^{(n+1)}(m)$ for some m.

note: $R_n(a) = f(a) - \underbrace{\sum_{k=0}^{n} \frac{(a-a)^k}{k!} f^{(k)}(a)}_{f(a)} = 0$

$$R_n'(a) = f'(a) - f'(a) = 0$$

$\vdots$

$R_n^{(n)}(a) = 0$ and $R_n^{(n+1)}(x) = f^{(n+1)}(x)$

and $S_n(a) = 0$

$$S_n'(a) = 0$$

$$\vdots$$

$$S_n^{(n)}(a) = 0$$

$$S_n^{(n+1)}(x) = 1$$

now apply the Cauchy thm.:

$$\frac{R_n(x)}{S_n(x)} = \frac{R_n(x) - \overbrace{R_n(a)}^{=0}}{S_n(x) - \underbrace{S_n(a)}_{=0}} = \frac{R_n'(m_1)}{S_n'(m_1)} \quad \text{for some } m_1 \in (a,x)$$

$$= \frac{R_n'(m_1) - R_n'(a)}{S_n'(m_1) - S_n'(a)} = \frac{R_n''(m_2)}{S_n''(m_2)} \quad \text{for some } m_2 \in (a, m_1)$$

$$= \ldots = \frac{R_n^{(n+1)}(m_{n+1})}{S_n^{(n+1)}(m_{n+1})} = \frac{f^{(n+1)}(m_{n+1})}{1} \quad \text{for some } m =: m_{n+1} \in (a,x)$$

$\square$