Jacobs University September 24, 2018
Fall 2018

Linear Algebra

Homework 3

Due on October 1, 2018

Problem 1 [**4 points**]

On the last homework sheet we considered a function that is homogeneous ($f(cx) = cf(x)$) but not additive (additive meaning that $f(x+y) = f(x) + f(y)$). Now we are looking for additive but non-homogeneous functions.

(a) Give an example of an additive function $f : \mathbb{C} \to \mathbb{C}$ ($\mathbb{C}$ is regarded as vector space over $\mathbb{C}$) which is not linear.

(b) Give an example of a non-linear but additive function $g : \mathbb{R} \to \mathbb{R}$, where $\mathbb{R}$ is regarded as vector space over $\mathbb{Q}$.

Problem 2 [**3 points**]

Given $f \in \mathcal{L}(V, W)$ for finite-dimensional vector spaces V, W and its matrix in chosen bases of V and W, what is the matrix of the dual map in the dual bases?

Problem 3 [**5 points**]

Prove the following properties of the dual map, for all $f, g \in \mathcal{L}(V, W)$, $h \in \mathcal{L}(W, X)$, $c \in F$, where all vector spaces V, W, X are finite dimensional.

(a) $(f+g)^* = f^* + g^*$,

(b) $(cf)^* = cf^*$,

(c) $(hf)^* = f^* h^*$,

(d) $1^* = 1$, $0^* = 0$,

(e) if V^{**} and W^{**} are canonically identified with V and W (as discussed in class), then $f^{**} : V^{**} \to W^{**}$ is canonically identified with $f : V \to W$.

Problem 4 [**4 points**]

For $f \in \mathcal{L}(V, W)$, prove that $\ker f$ and $\mathrm{im} f$ are indeed subspaces.

Problem 5 [**4 points**]

Let us consider the Pauli matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Prove the following properties:

(a) $[\sigma_\ell, \sigma_m] = 2i\varepsilon_{\ell mn}\sigma_n$, where ℓ, m, n can be $1, 2, 3$, $\varepsilon_{\ell mn}$ is 1 for even permutations and -1 for odd ones, and $[A, B] := AB - BA$,

(b) $\sigma_\ell\sigma_m + \sigma_m\sigma_\ell = 2\delta_{\ell m}\sigma_0$,

(c) the matrices $i\sigma_1$, $i\sigma_2$, $i\sigma_3$ form a basis of the space of complex 2×2 matrices with trace zero, and the matrices σ_0,$i\sigma_1$,$i\sigma_2$,$i\sigma_3$ form a basis of the space of complex 2×2 matrices.