Jacobs University
Fall 2018

October 8, 2018

Linear Algebra

Homework 5

Due on October 15, 2018

Problem 1 [**2 points**]
Let $M, N \subset L$ be subspaces. Prove that the mapping

$$(M+N)/N \to M/(M \cap N), m+n+N \mapsto m + M \cap N$$

is a linear isomorphism.

Problem 2 [**2 points**]
Prove that the canonical mapping

$$M \to (M \oplus N)/N, m \mapsto m + N$$

is an isomorphism.

Problem 3 [**10 points**]
Let M be a subspace of L.

(a) Prove that $\dim M + \dim M^{\perp} = \dim L$.

(b) Prove that $(M^{\perp})^{\perp}$ as a subspace of L^{**} is canonically isomorphic to M (under the canonical isomorphism that identifies elements of L^{**} with elements of L).

(c) In class we discussed a big diagram claiming many isomorphisms. Prove all of these! Half a point for each. (*Hint: Each one follows quite easily from applying the lemmas discussed in class and part (a) and (b).*)

Problem 4 [**3 points**]
For $f \in \mathcal{L}(L, M)$, prove that $f(x) = y$ has a solution if and only if y is orthogonal to the kernel of the dual map $f^* : M^* \to L^*$.

Problem 5 [**3 points**]
Prove that the column rank of a matrix (the maximal number of linearly independent column vectors) equals its row rank (the maximal number of linearly independent row vectors). Proceed by using that if a map $f : L \to M$ is represented in some basis by a matrix A, then the dual map in the dual basis is represented by the transposed matrix A^t.