Session 5
Sep. 19, 2018

Bijective maps:

Let $f \in \mathcal{L}(V,W)$ be bijective. Then the inverse $f^{-1}: W \to V$ exists.

Is f^{-1} linear?

choose $w_1 \neq w_2 \in W$, then $\exists v_1, v_2 \in V$ s.t. $f(v_1) = w_1$, $f(v_2) = w_2$.

Then $f(v_1+v_2) = f(v_1) + f(v_2)$ and $f(cv_1) = cf(v_1)$ $\forall c \in F$

apply f^{-1}: $f^{-1}(f(v_1+v_2)) = f^{-1}(f(v_1)+f(v_2))$ and $f^{-1}f(cv_1) = cv_1 = f^{-1}(cw_1)$
$= cf^{-1}(w_1)$

$$\Rightarrow \underbrace{v_1+v_2}_{f^{-1}(w_1)+f^{-1}(w_2)} = f^{-1}(w_1+w_2)$$

Ex.: • $f: \mathrm{Pol}(\mathbb{R}) \to \mathrm{Pol}(\mathbb{R})$, $g(x) \mapsto x^2 g(x)$ (mult. with x^2)

is injective but not surjective

• backward shift on F^∞: $(x_1, x_2, x_3, \dots) \mapsto (x_2, x_3, \dots)$

not injective but surjective

Bijective $f \in \mathcal{L}(V,W)$ are called isomorphisms.

If there exists an isomorphism between V and W, then V and W are called isomorphic, and we write $V \cong W$.

Thm.: Let V and W be finite dimensional vector spaces. Then they are isomorphic if and only if $\dim V = \dim W$.

Proof: "$\Rightarrow$" Let $f: V \to W$ be an isomorphism, choose basis $\{v_1, \dots, v_n\}$ in V.

Claim: $\{f(v_1), \dots, f(v_n)\}$ is a basis of W.

check lin. indep.: $\sum_{i=1}^n c_i f(v_i) = 0$

$$\Rightarrow f\left(\sum_{i=1}^n c_i v_i\right) = 0 \Rightarrow f^{-1} f\left(\sum_{i=1}^n c_i v_i\right) = \sum_{i=1}^n c_i v_i = f^{-1}(0) = 0$$

(lin. of f^{-1})

$\Rightarrow$ all c_i's are zero ✓

generating? choose $w \in W$

$$\text{let } f^{-1}(w) = v = \sum_{i=1}^n c_i v_i \overset{f^{-1}}{\Rightarrow} w = f(v) = f\left(\sum_{i=1}^n c_i v_i\right) = \sum_{i=1}^n c_i f(v_i) \checkmark$$

$\Rightarrow \dim V = \dim W$

"$\Leftarrow$" suppose $\dim V = \dim W = n$

choose a basis $\{v_1, \dots, v_n\}$ of V and $\{w_1, \dots, w_n\}$ of W.

According to previous lemma $\exists$ an isomorphism $f: V \to W$ s.t. $f(v_i) = w_i$.

i.e., $f\left(\sum_{i=1}^n c_i v_i\right) = \sum_{i=1}^n c_i f(v_i) = \sum_{i=1}^n c_i w_i$

f is bijective, since $f^{-1}\left(\sum_{i=1}^n c_i w_i\right) := \sum_{i=1}^n c_i v_i$ $\square$

Usually (except if $V=W=\{0\}$, and $\dim V=\dim W=1$ with $\mathbb{F}=\{0,1\}$) there are many isomorphisms.

Isomorphisms that do not depend on "arbitrary choices" (e.g., basis) are called canonical or natural isomorphisms.

Otherwise they are called non-canonical or accidental.

<u>Ex. of accidental isomorphism:</u>

consider $\dim V = n$ and maps $f: V \to V^*$, choose basis $\{v_1, \dots, v_n\}$ of V, def. isomorphism f by $f(v_i) = v_i^*$, where $\{v_1^*, \dots, v_n^*\}$ is the dual basis (basis of V^*).

But f is different if we choose different basis:

Ex.: $\dim V = 1$, basis $\{v_1\}$ of V for any $V \ni v_1 \neq 0$, dual basis $\{v_1^*\}$, and isomorphism f, s.t. $f(v_1) = v_1^*$ $\quad (v_1^*(v_1) = 1)$

other basis: $\{cv_1\}$, $c \neq 0, |c| \neq 1 \Rightarrow$ dual basis: $\{c^{-1}v_1^*\}$
$c \in \mathbb{F}$

$$(c^{-1}v_1^*(cv_1) = c^{-1}c\underbrace{v_1^*(v_1)}_{=1} = 1)$$

isomorphism: $\tilde{f}(cv_1) = c^{-1}v_1^*$

or $\tilde{f}(v_1) = c^{-2}v_1^*$

<u>Ex. of canonical isomorphism:</u>

consider $\dim V = n$ and isomorphism $V \to (V^*)^* = V^{**}$ (double dual) double dual contains lin. fct.s from V^* to $\mathbb{F}$

def. isomorphism $\underset{\text{see below}}{\varepsilon : V \to \underbrace{V^{**}}}$, $v \mapsto v^{**} = \underbrace{[\overset{\in V^*}{\overbrace{f}} \mapsto \overset{\in \mathbb{F}}{\overbrace{f(v)}}]}_{V^* \to \mathbb{F}}$

or $\varepsilon(v) = v^{**}$ with $\underbrace{v^{**}(f)}_{\in \mathbb{F}} := \underbrace{f(v)}_{\in \mathbb{F}}$ $\quad (f \in V^*)$

now: for each v, $\varepsilon(v)$ is linear (check that indeed ε maps into V^{**})

take $f_1, f_2 \in V^*$, $c_1, c_2 \in \mathbb{F}$, then $\underbrace{v^{**}}_{\varepsilon(v)}(c_1 f_1 + c_2 f_2) := (c_1 f_1 + c_2 f_2)(v)$

linear

$= c_1 f_1(v) + c_2 f_2(v)$

$= c_1 v^{**}(f_1) + c_2 v^{**}(f_2)$

also ε is linear, since $\big(\varepsilon(c_1 v_1 + c_2 v_2)\big)(f) = f(c_1 v_1 + c_2 v_2)$

$\overset{f \text{ lin.}}{=} c_1 f(v_1) + c_2 f(v_2)$

$= c_1 \big(\varepsilon(v_1)\big)(f) + c_2 \big(\varepsilon(v_2)\big)(f)$

Is ε an isomorphism?

Let $\{v_1, \dots, v_n\}$ be basis of V, $\{v_1^*, \dots, v_n^*\}$ basis of V^* (dual basis to V), $\{v_1^{**}, \dots, v_n^{**}\}$ be basis of V^{**} (dual basis to V^*).

$\Rightarrow$ then the map $\sum_{i=1}^{n} c_i v_i \longrightarrow \sum_{i=1}^{n} c_i v_i^{**}$ is an isomorphism (clear)

this map is ε, since

dual basis: $v_i^*(v_j) = \delta_{ij}$

$\varepsilon(v_j)\left(\sum_{i=1}^{n} c_i v_i^*\right) = \left(\sum_{i=1}^{n} c_i v_i^*\right)(v_j) = c_j = v_j^{**}\left(\sum_{i=1}^{n} c_i v_i^*\right)$

dual basis: $v_j^{**}(v_i^*) = \delta_{ij}$

$\Rightarrow \varepsilon(v_j) = v_j^{**}$

$\Rightarrow \varepsilon$ is an isomorphism that does not depend on basis choice.

Dual map:

Let $f: V \to W$ be linear. Then dual map $f^*: W^* \to V^*$ is def. by

$$\underbrace{f^*(w^*)(v)}_{\in F} = \underbrace{w^*(f(v))}_{\in F} \quad \forall v \in V.$$