Session 19
Nov. 21, 2018

Thm.: a) symplectic over any field F, symmetric over $\mathbb{C}$:
up to isometry determined by n, r_0

b) Inertia thm. (or Sylvester's law of inertia): symmetric over $\mathbb{R}$, Hermitian over $\mathbb{C}$:
up to isometry determined by (r_0, r_+, r_-), independent of choice of orthogonal decomposition

Remarks:
- all non-degenerate symplectic forms on spaces with same dimension are isometric (same for non-deg. symm. on $\mathbb{C}$)
- non-degenerate symplectic spaces have even dimension
- why do we study only the above symmetries?
  ↳ exactly these preserve orthogonality (see HW)
  (note: g Hermitian $\Rightarrow \bar{g}^T = g \Rightarrow \overline{(ig)}^T = -i\,\bar{g}^T = -(ig)$, i.e., ig is Hermitian antisymm.)
  ↳ any g can be uniquely decomposed into symm. + antisymm. (see HW)

Proof:

a) let $L = \bigoplus_{i=1}^{m} L_i$, L_i pairwise orthogonal, 1-dim. for symm., 1 or 2-dim. for symplectic.

Step 1: we show $\ker g$ = sum of 1-dim. degenerate L_i, say these are $L_1, \dots, L_k$

- $\bigoplus_{i=1}^{k} L_i \subset \ker g$, since $g(\sum_{i=1}^{k} \underbrace{e_i^0}_{\in L_i}, \sum_{j=1}^{n} \underbrace{e_j}_{\in L_j}) = \sum_{i \neq j} \underbrace{g(e_i^0, e_j)}_{=0,\text{ since } L_i \text{ orthogonal to } L_j \text{ for } i\neq j} + \sum_{i=1}^{k} g(e_i^0, e_i) = 0$

since also $g(e_i^0, e_i) = 0$, since $L_1, \dots, L_k$ degenerate

• let $L_0 := \bigoplus_{i=1}^{k} L_i$ and consider $\ell = \sum_{i=1}^{n} \underbrace{\ell_i}_{\in L_i}$ s.t. $\exists j > k$ with $\ell_j \neq 0$

we show that then $\ell \notin \ker g$, i.e., $\ker g \subset L_0$

↳ orthogonal: $g(\ell, \ell_j) = g(\ell_j, \ell_j) \neq 0$, because L_j non-degenerate

↳ symplectic: $\exists \ell_j' \in L_j$ s.t. $g(\ell, \ell_j') = g(\ell_j, \ell_j') \neq 0$, bc. L_j non-degenerate

$\Rightarrow \bigoplus_{i=1}^{k} \underbrace{L_i}_{1\text{-dim.}} = \ker g$ and $r_0 = \dim \ker g = k$

Step 2: Isometry for same n, r_0:

consider (L, g) and (L', g') with same n and r_0, let $L = \bigoplus_{i=1}^{m} L_i$ such that $\ker g = \bigoplus_{i=1}^{r_0} L_i$, let $L' = \bigoplus_{i=1}^{m} L_i'$ such that $\ker g' = \bigoplus_{i=1}^{r_0} L_i'$.

Let $f_i : L_i \to L_i'$ be the isometries from II.2, then $\bigoplus f_i$ is isometry between (L, g) and (L', g').

b) (L, g) and (L', g') orthogonal spaces over $\mathbb{R}$ or Hermitian over $\mathbb{C}$, signatures

- (r_0, r_+, r_-) for decomposition $L = \bigoplus L_i$
- (r_0', r_+', r_-') for decomposition $L' = \bigoplus L_i'$

step 1: isometric $\Rightarrow$ signatures are the same

step 2: same signature $\Rightarrow$ isometric

Step 1:

• isometric $\Rightarrow \dim L = \dim L'$, so $r_0 + r_+ + r_- = r_0' + r_+' + r_-'$

- same argument as part a): $r_0 = \dim \ker g$, $r_0' = \dim \ker g'$

so isometry $\Rightarrow r_0 = r_0' \Rightarrow r_+ + r_- = r_+' + r_-'$

- Let $L = \underbrace{L_0}_{\text{sum of null spaces}} \oplus \underbrace{L_+}_{\text{sum of pos. spaces}} \oplus \underbrace{L_-}_{\text{sum of neg. spaces}}$, $L' = L_0' \oplus L_+' \oplus L_-'$

we show that $r_+ = \dim L_+ > r_+' = \dim L_+'$ leads to contradiction (same argument for $r_+ < r_+'$), which will conclude step 1

choose $l \in L_{+}$, then $f(l) = \underbrace{f(l)_0}_{\in L_0'} + \underbrace{f(l)_+}_{\in L_+'} + \underbrace{f(l)_-}_{\in L_-'}$

note: $L_+ \to L_+'$, $l \mapsto f(l)_+$ is linear

$\dim L_+ > \dim L_+' \Rightarrow \exists\, 0 \neq l \in L_+$ s.t. $f(l)_+ = 0$ (non-trivial kernel), i.e.

$f(l) = f(l)_0 + f(l)_-$

now $g(l,l) > 0$, so also $g'(f(l), f(l)) > 0$ (f isometry), but

$$g'(f(l), f(l)) = g'(\underbrace{f(l)_0}_{\in L_0'} + f(l)_-, f(l)_0 + f(l)_-)$$
$$= g'(\underbrace{f(l)_-}_{\in L_-'}, f(l)_-) < 0 \Rightarrow \text{contradiction}$$

<u>Step 2:</u> let (L, g) and (L', g') have same signature.

As before let $f_i : L_i \to L_i'$ be the isometries from from II.2 (each f_i preserves sign)

$\Rightarrow \bigoplus f_i : L \to L'$ is isometry

$\square$

Implications:

- orthogonal and Hermitian spaces have an orthogonal basis, i.e., basis $\{e_1, \dots, e_n\}$

  with $g(e_i, e_j) = 0$ for $i \neq j$ (choose any $e_i \in L_i$ when $L = \bigoplus_{i=1}^{n} \underbrace{L_i}_{\text{1-dim. and orthogonal}}$)

  they even have orthonormal basis, i.e., additionally $g(e_i, e_i) = \begin{cases} +1 & \text{or} \\ 0 & \text{or} \\ -1 \end{cases}$

  $\Rightarrow$ Gram matrix $\begin{pmatrix} E_{r_+} & & 0 \\ & -E_{r_-} & \\ 0 & & 0_{r_0} \end{pmatrix}$

  non-degenerate case: $g(l, e_i) = g(l', e_i)\ \forall i \Rightarrow g(l - l', e_i) = 0\ \forall i \Rightarrow l = l'$

  so we can write any $l \in L$ as $l = \sum_{i=1}^{n} \frac{g(l, e_i)}{g(e_i, e_i)} e_i$ ($g(l, e_i)$ on lhs and rhs same)

- symplectic spaces have symplectic basis $\{e_1, \dots, e_r, \tilde{e}_1, \dots, \tilde{e}_r, e'_1, \dots, e'_{n-2r}\}$

  with $g(e_i, \tilde{e}_i) = -g(\tilde{e}_i, e_i) = 1$ and all other combinations $= 0$

  $\Rightarrow$ Gram matrix $\begin{pmatrix} 0 & E_r & 0 \\ -E_r & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$