Session 20
Nov. 26, 2018

- quadratic forms: let $h: L \times L \to F$ be bilinear form, then

  $q: L \to F$, $q(l) = h(l,l)$ is called a quadratic form

  a symm. bilinear form g s.t. $q(l) = g(l,l)$ is called polarization of q

  we can just set $g(l,m) = \frac{1}{2}\left(h(l,m) + h(m,l)\right)$ (symm. and $g(l,l) = q(l)$)

  this g is unique $\left(q(l) = g_1(l,l) = g_2(l,l) \Rightarrow (g_1 - g_2)(l,l) = 0 \Rightarrow g_1 - g_2 = 0\right)$

  in terms of q: $g(l,m) = \frac{1}{2}\left(g(l+m, l+m) - g(l,l) - g(m,m)\right)$

  $$= \frac{1}{2}\left(q(l+m) - q(l) - q(m)\right)$$

  $\Rightarrow$ correspondence for orth. geometries: $(L, \underset{\substack{\uparrow \\ \text{symm. bilinear}}}{g}) \longleftrightarrow (L, \underset{\substack{\uparrow \\ \text{quadratic form}}}{q})$

- orthogonalization algorithm, orthogonal polynomials

- application to Euclidean and unitary spaces

- orthogonal and unitary operators (the isometries of Euclidean/unitary spaces)

- self-adjoint operators (diagonalizable: spectral thm.) and quantum mechanics

II.4 Orthogonalization Algorithms

How do we turn quadratic form $q(\underbrace{x_1,\dots,x_n}_{\text{coordinates in some basis}}) = \sum_{i,j=1}^{n} a_{ij} x_i x_j \quad (a_{ij} = a_{ji})$

into form $q(z_1,\dots,z_n) = \sum_{i=1}^{n} b_i z_i^2$ (by change of basis)?

say $a_{11} \neq 0$ ($a_{11} = 0$ treated slightly different):

↳ write $q(x_1,\dots,x_n) = a_{11} x_1^2 + x_1(2a_{12}x_2 + \dots + 2a_{1n}x_n) + q'(x_2,\dots,x_n)$

$$= a_{11}\Big(\underbrace{x_1 + \frac{a_{12}}{a_{11}} x_2 + \dots + \frac{a_{1n}}{a_{11}} x_n}_{Y_1}\Big)^2 + q''(\underbrace{x_2,\dots,x_n}_{Y_2,\dots,Y_n})$$

$\Rightarrow q(Y_1,\dots,Y_n) = a_{11} Y_1^2 + q''(Y_2,\dots,Y_n)$, then repeat for $Y_2,\dots,Y_n$

$\Rightarrow q(z_1,\dots,z_n) = \sum_{i=1}^{n} b_i z_i^2$

with $u_i = \begin{cases} \sqrt{|b_i|}\, z_i, & b_i \in \mathbb{R} \\ \sqrt{b_i}\, z_i, & b_i \in \mathbb{C} \end{cases}$, coefficients become $0, \pm 1$ (canonical form)

Gram-Schmidt orthogonalization:

(L, g) orth. or Hermitian, given in some basis $\{e_1', \dots, e_n'\}$

let $L_i' = \operatorname{span}(e_1', \dots, e_i')$, and $L_1', \dots, L_n'$ non-degenerate. Then there is a unique (up to scaling) orthogonal basis $\{e_1, \dots, e_n\}$ with $\operatorname{span}(e_1, \dots, e_i) = L_i' \ \forall i$.

(induction: $L_i = L_{i-1} \oplus L_{i-1}^{\perp}$ and $\dim L_{i-1}^{\perp} = \dim L_i - \dim L_{i-1} = 1$).

Explicit algorithm:

- start with $e_1 = e_1'$
- say $e_1, \dots, e_{i-1}$ already given

then look for $e_i = e_i' - \sum_{j=1}^{i-1} \gamma_j e_j \quad (\gamma_j \in \mathbb{F})$

need $g(e_i, e_j) = 0 \; \forall j = 1, \dots, i-1 \Rightarrow 0 = g(e_i', e_j) - \gamma_j g(e_j, e_j)$

$$\Rightarrow \gamma_j = \frac{g(e_i', e_j)}{g(e_j, e_j)}$$

$$\Rightarrow e_i = e_i' - \sum_{j=1}^{i-1} \frac{g(e_i', e_j)}{g(e_j, e_j)} e_j$$

note: for orthonormal additionally divide by $\begin{cases} g(e_i, e_i)^{\frac{1}{2}} & \text{in } \mathbb{C} \\ |g(e_i, e_i)|^{\frac{1}{2}} & \text{in } \mathbb{R} \end{cases}$

next: orthogonal polynomials

consider some space of fct.s $f: (a,b) \to \begin{cases} \mathbb{R} \\ \mathbb{C} \end{cases}$ and bilinear (sesquilinear) forms

$$g(f_1, f_2) = \int_a^b f_1(x) \overline{f_2(x)}\, \sigma(x) dx \quad , \; \sigma: \text{weight fct. (often } \sigma \geq 0)$$

Want: approximation of f by $\sum_i a_i f_i$, $\{f_i\}$ orthogonal (or orthonormal) fct.s

Ex.:

- Fourier series: $\sigma(x) = 1$, $(a,b) = (0, 2\pi)$

recall from Advanced Calculus: $f_k = \frac{1}{\sqrt{2\pi}} e^{ikx}$, $k \in \mathbb{Z}$ orthonormal

$$\Rightarrow f(x) = \sum_{k \in \mathbb{Z}} c_k f_k \text{ with } c_k = g(f, f_k) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x) e^{-ikx} dx$$

- $\sigma(x)=1$, $(a,b)=(-1,1)$

Then Gram-Schmidt applied to $\{1, x, x^2, \dots\}$ gives Legendre polynomials

$$P_0(x)=1, \quad P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n, \quad n\geq 1 \qquad \text{(exercise)}$$

(they appear, e.g., when solving $\Delta\varphi=0$ (Laplace eq.) in spherical coordinates)

- Chebyshev polynomials: $\sigma(x)=\frac{1}{\sqrt{1-x^2}}$, $(a,b)=(-1,1)$

- Hermite polynomials: $\sigma(x)=e^{-x^2}$, $(a,b)=(-\infty,\infty)$

(eigenstates of harmonic oscillator)