II.7 Self-adjoint Operators

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L finite dim. Euclidean (unitary) space with scalar product $\langle \cdot, \cdot \rangle$

Def.: $f: L \to L$ with $\langle f(e_1), e_2 \rangle = \langle e_1, f(e_2) \rangle \ \forall e_1, e_2 \in L$ is called **self-adjoint**.

Lemma: let $f: L \to L$ be diagonalizable in some orthonormal basis with real eigenvectors.

Then f is self-adjoint.

Proof: $\{e_1, \dots, e_n\}$ ONB, $f(e_j) = \lambda_j e_j$, $\lambda_j \in \mathbb{R}$ eigenvectors

$$\Rightarrow \langle f(\underbrace{\sum_{i=1}^n x_i e_i}_{e_1}), \underbrace{\sum_{j=1}^n y_j e_j}_{e_2} \rangle = \sum_{i=1}^n \lambda_i x_i \overset{(-)}{y_i}$$

$$\langle \sum x_i e_i, f(\sum_j y_j e_j) \rangle = \sum_{i=1}^n \underbrace{\overline{\lambda_i}}_{= \lambda_i \in \mathbb{R}} x_i \overset{(-)}{y_i} \qquad \square$$

later: converse also true

Ex.: quantum mechanics

↳ associate measurement results with real numbers → write them on diagonal

→ a self-adjoint operator describes possible measurement outcomes

↳ $\psi_t \in$ Hilbert space describes dynamics

↳ dynamics: $i \frac{d}{dt} \psi_t = \underbrace{H}_{= \text{Hamiltonian (self-adjoint)}} \psi_t$, formally solved by $\psi_t = \underbrace{e^{-iHt}}_{\text{unitary operator}} \psi_0$

Adjoint operator:

recall: $f: L \to L$, dual map $f^*: \overset{(-)}{L}{}^* \to \overset{(-)}{L}{}^*$ s.t. $f^*(m^*)\ell = m^*(f(\ell))$

let g be scalar product, def. via isomorphism $\tilde{g}: L \to \overset{(-)}{L}{}^*$, i.e., $g(\ell, m) = \tilde{g}(\ell)(m)$

$\tilde{g}^{-1}: \overset{(-)}{L}{}^* \to L$ identifies elements of (conjugated) dual space with L!

$$\Rightarrow g(\ell, f(m)) = \tilde{g}(\ell)(f(m)) = f^*(\tilde{g}(\ell))(m) = \tilde{g}(\tilde{g}^{-1} f^* \tilde{g})(\ell)(m)$$
$$= g((\tilde{g}^{-1} f^* \tilde{g})(\ell), m)$$

$\tilde{g}^{-1} f^* \tilde{g}$ called adjoint, usually denoted f^* again (or f^+ on unitary spaces)

so f self-adjoint $\Leftrightarrow f^* = f$

note: · if matrix of f in some ONB is A, then matrix of f^* is $\overset{(-)}{A}{}^T$

· f called symmetric (Hermitian) if matrix in some ONB symm. (Hermitian)

· if $\langle \cdot, \cdot \rangle$ is a symm. (Hermitian) scalar product, so is $\langle \cdot, \cdot \rangle_f = \langle f(\cdot), \cdot \rangle$

$$\Rightarrow \overline{\langle \ell_2, \ell_1 \rangle_f} = \overline{\langle f(\ell_2), \ell_1 \rangle} = \overline{\langle \ell_2, f^*(\ell_1) \rangle} = \langle f^*(\ell_1), \ell_2 \rangle = \langle \ell_1, \ell_2 \rangle_{f^*}$$

$\Rightarrow$ self-adjoint operators $\Leftrightarrow$ symm. (Hermitian) scalar products

Now: converse statement:

Theorem:

a) $f: L \to L$ self-adjoint $\Leftrightarrow f$ diagonalizable in some ONB with real eigenvectors
(Spectral Theorem)

b) If $f: L \to L$ self-adjoint, then eigenvectors for different eigenvalues are orthogonal

<u>Proof:</u> a) "$\Leftarrow$" Lemma above

"$\Rightarrow$"

- show all eigenvalues $\in \mathbb{R}$:

  $\hookrightarrow$ f Hermitian, $f(e) = \lambda e \Rightarrow \bar{\lambda} \langle e, e \rangle = \langle e, f(e) \rangle = \langle f(e), e \rangle = \lambda \underbrace{\langle e, e \rangle}_{>0}$

  $\hookrightarrow$ f symmetric: consider $L^{\mathbb{C}}$ (complexification, see I.10) with Hermitian

  scalar product $\langle e_1 + i e_2, e_3 + i e_4 \rangle = \langle e_1, e_3 \rangle + \langle e_2, e_4 \rangle + i \langle e_2, e_3 \rangle - i \langle e_1, e_4 \rangle$

  $\Rightarrow L^{\mathbb{C}}$ unitary space, $f^{\mathbb{C}}$ Hermitian (check)

  since matrices of f and $f^{\mathbb{C}}$ the same, eigenvalues the same, in particular real

- induction in $\dim L$:

  $\hookrightarrow$ $\dim L = 1$ clear

  $\hookrightarrow$ $\dim L > 1$: choose some eigenvalue λ, $f(e_\lambda) = \lambda e_\lambda$, $L_\lambda = \operatorname{span}\{e_\lambda\}$

  $\Rightarrow L = L_\lambda \oplus L_\lambda^\perp$ (see II.3)

  now $\langle e_\lambda, f(\underbrace{e_\lambda^\perp}_{\in L_\lambda^\perp}) \rangle = \langle f(e_\lambda), e_\lambda^\perp \rangle = \lambda \langle e_\lambda, e_\lambda^\perp \rangle = 0$

  so $L_\lambda^\perp$ f-invariant

  by induction hypothesis $f|_{L_\lambda^\perp}$ diagonalized, so f diagonalized in the ONB of $L_\lambda^\perp$ and $\frac{e_\lambda}{\|e_\lambda\|}$.

b) $f(e_1) = \lambda_1 e_1$, $f(e_2) = \lambda_2 e_2$, $\lambda_1 \neq \lambda_2$

$\Rightarrow \lambda_1 \langle e_1, e_2 \rangle = \langle f(e_1), e_2 \rangle = \langle e_1, f(e_2) \rangle = \lambda_2 \langle e_1, e_2 \rangle \Rightarrow \langle e_1, e_2 \rangle = 0$

$\square$

Applications:

- A Hermitian $\mapsto U(A) = e^{iA}$ is surjective

  (any unitary U can be diagonalized to form $\begin{pmatrix} e^{i\varphi_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\varphi_n} \end{pmatrix}$

  $\Rightarrow$ A in this basis is $\begin{pmatrix} \varphi_1 & & 0 \\ & \ddots & \\ 0 & & \varphi_n \end{pmatrix}$ $\Rightarrow$ change basis back to get original A)

- $f: \mathbb{R}^n \to \mathbb{R}$, then Hessian $\frac{\partial^2 f}{\partial x_i \partial x_j}$ is symmetric $\to$ analyse critical points via eigenvalues

note:

- more general: normal operators $:= \{ f : f f^* = f^* f \}$

  $= \{ f : f \text{ diagonalizable in some ONB} \}$

  similar proof strategy as in thm. above

Outlook: three versions of the spectral thm.:

Let f be Hermitian.

- $A_f = B D_f B^*$, D_f diagonal or $f|_{c_i} = \lambda_i \, \mathrm{id}$

- spectral decomposition: $f = \sum_i \lambda_i p_i$ (if f diagonal in ONB $\{e_i\}$, then $p_i =$ orth. projection on $\operatorname{span}\{e_i\}$)

- existence of functional calculus: $G(f)$ well-defined for G in some class of functions

(later in functional analysis: do this for
- compact op.s (limits of matrices)
- continuous (or bounded) op.s ($\|fx\| \leq c\|x\|$)
- unbounded op.s (like derivatives, ...)