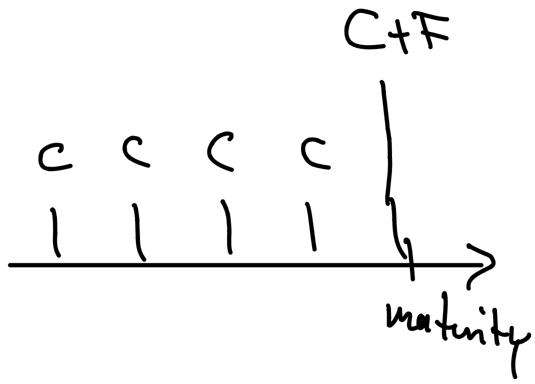
Session 4
Sep. 14, 2018

1.3 Bonds

contract: issuer = borrower pays interest and principal
to bondholder = lender

↳ usually for long-term debts

↳ repaid at maturity date



Level coupon bond:

$$P = \sum_{i=1}^{n\cdot m} \frac{C}{(1+\frac{r}{m})^i} + \frac{F}{(1+\frac{r}{m})^{n\cdot m}}$$

C = coupon payment

r = interest rate

F = par value

n = # of periods (usually years)

m = # of compoundings per period

$C = \frac{F\cdot c}{m}$, c = coupon rate $\Rightarrow P = F\left(\sum_{i=1}^{nm} \frac{c/m}{(1+\frac{r}{m})^i} + \frac{1}{(1+\frac{r}{m})^{nm}}\right)$

Ex.: 20 year, $\underbrace{9\%}_{\text{coupon rate} = c}$ bond, $\underbrace{\text{semiannual}}_{\to m=2}$ compounding, interest rate $r = 8\%$

$\Rightarrow$ price $P = F\left(\sum_{i=1}^{40} \frac{0.045}{(1.04)^i} + \frac{1}{(1.04)^{40}}\right) = 1.099 \cdot F$

$\Rightarrow$ bond is sold at 109.9% of par

(e.g., par value $F = 1000\$ \Rightarrow P = 1099\$$ and $C = 45\$$)

remember: $\sum_{i=0}^{n} x^i = \frac{1-x^{n+1}}{1-x}$, $x = \frac{1}{1+y}$

$$\sum_{i=1}^{n} \frac{1}{(1+y)^i} = -1 + \frac{1-(1+y)^{-n-1}}{1-\frac{1}{1+y}} = -1 + \left(\frac{1-(1+y)^{-n-1}}{y}\right)(1+y)$$

$$= -1 + \left(\frac{1+y-(1+y)^{-n}}{y}\right)$$

$$= \frac{1-(1+y)^{-n}}{y}$$

take $m=1$:

$$P = F\left(\sum_{i=1}^{n} \frac{c}{(1+y)^i} + \frac{1}{(1+y)^n}\right) \quad , c = \text{coupon rate}$$

$$= F\left(c\left(\frac{1-(1+y)^{-n}}{y}\right) + (1+y)^{-n}\right)$$

$$= F\left(\frac{c+(y-c)(1+y)^{-n}}{y}\right)$$

$$= F\left(\frac{c}{y} + \frac{\left(1-\frac{c}{y}\right)}{(1+y)^n}\right) \quad \left(m>1:\ (1+y)^n \longrightarrow \left(1+\frac{y}{m}\right)^{nm}\right)$$

- $c = y$, then "bond sells at par"
- $c > y$, then "bond sells above par" or "at a premium"
- $c < y$, then "bond sells below par" or "at a discount"

Zero-coupon bonds: $C=0 \to$ single payment F

$$\to P = \frac{F}{(1+\frac{y}{m})^{nm}}$$

terminology: yield to maturity $= IRR = y$, given C, F, P

1.4 Immunization

reduce risk coming from changes in the interest rate y if future liability has to be met.

say liability L at period m (m called "horizon" here)

consider zero-coupon bonds ($C=0$) with maturity n, par value F:

$$FV_m = (1+y)^m P$$
$$= (1+y)^m \frac{F}{(1+y)^n}$$
$$= F (1+y)^{m-n}$$

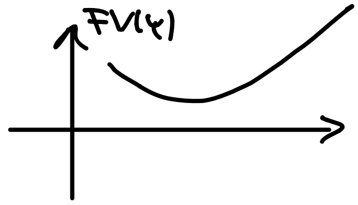
portfolio with two zero-coupon bonds with $n_1 < m$, F_1 and $n_2 > m$, F_2:

$$FV_m = F_1 (1+y)^{m-n_1} + F_2 (1+y)^{m-n_2} \overset{!}{=} L \quad \text{to meet liability}$$

also want stability w.r.t. changes in y $\Rightarrow$ find minimum of $FV_m(y)$

with respect to

$\rightarrow$ look for $\dfrac{\partial FV_m(y)}{\partial y} = 0$



stable

$$\Rightarrow F_1 (m-n_1) (1+y)^{m-n_1-1} + F_2 (m-n_2)(1+y)^{m-n_2-1} = 0$$

$$\Rightarrow F_1 (m-n_1) (1+y)^{-n_1} + F_2 (m-n_2)(1+y)^{-n_2} = 0$$

$$\Rightarrow m \underbrace{\frac{F_1}{(1+y)^{n_1}}}_{P_1} + m \underbrace{\frac{F_2}{(1+y)^{n_2}}}_{P_2} = n_1 \underbrace{\frac{F_1}{(1+y)^{n_1}}}_{P_1} + n_2 \underbrace{\frac{F_2}{(1+y)^{n_2}}}_{P_2}$$

$$\Rightarrow P = P_1 + P_2 \Rightarrow mP = n_1 P_1 + n_2 P_2$$

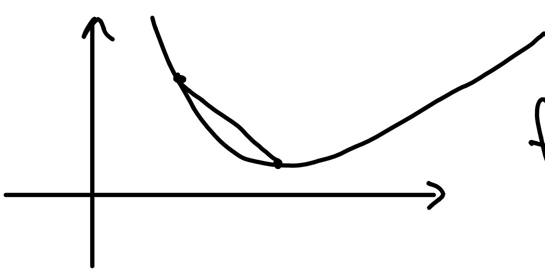
$$\Rightarrow m = \underbrace{\frac{1}{P}(n_1 P_1 + n_2 P_2)}_{=: MD = \text{Macaulay duration}} \rightarrow \text{weighted average}$$

note: $m = (1+y) \underbrace{\left(-\frac{1}{P}\frac{\partial P}{\partial y}\right)}_{\text{price volatility}}$ $\qquad P = \dfrac{F_1}{(1+y)^{n_1}} + \dfrac{F_2}{(1+y)^{n_2}}$

gen. formula: want MD to match horizon m

to have a minimum, we need FV(y) to be convex (for a certain range of y)

Convex:



$$f(\lambda x_1 + (1-\lambda) x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$
$$\forall \lambda \in [0,1]$$

$$\frac{\partial^2 FV(y)}{\partial y^2} = F_1 \underbrace{(m-n_1)(m-n_1-1)}_{\geq 0}(1+y)^{m-n_1-2} + F_2 \underbrace{(m-n_2)(m-n_2-1)}_{\geq 0}(1+y)^{m-n_2-2}$$

$\Rightarrow$ minimum

general immunization conditions:

(1) $FV = L$ at horizon m

(2) $\frac{\partial FV(y)}{\partial y} = 0$, or $MD = m$

(3) $FV(y)$ convex around relevant y

a few remarks:

- gen. cash flows $\sum_{i=1}^{n} \frac{C_i}{(1+y)^i}$

Macaulay duration $MD = \frac{1}{P} \sum_{i=1}^{n} \frac{i C_i}{(1+y)^i} = (1+y)\left(-\frac{1}{P}\frac{\partial P}{\partial y}\right)$

- for level-coupon bond : $P = \sum_{i=1}^{n} \frac{C}{(1+y)^i} + \frac{F}{(1+y)^n}$, $C = c \cdot F$
$(m=1)$

volatility $-\frac{1}{P}\left(\frac{\partial P}{\partial y}\right) = \dfrac{\frac{c}{y} n - \frac{c}{y^2}(1+y)\left((1+y)^n - 1\right) - n}{\frac{c}{y}(1+y)\left((1+y)^n - 1\right) + (1+y)}$

$$MD = \frac{c(1+y)\left((1+y)^n - 1\right) + ny(y-c)}{cy\left((1+y)^n - 1\right) + y^2}$$