Jacobs University | September 5, 2019
Fall 2019

Calculus on Manifolds

Homework 1

Due on September 12, 2019

Problem 1 [**4 points**]
Let $M_{n\times n}(\mathbb{R})$ denote the set of real $n \times n$ matrices. We consider the function

$$f : M_{n\times n}(\mathbb{R}) \to \mathbb{R}, f(A) = \det(\mathbb{1} + A).$$

Show that f is differentiable and $Df(0)(H) = \mathrm{Tr}(H) := \sum_{i=1}^n H_{ii}$ (the trace of H).

Problem 2 [**6 points**]
For fixed integer n, we consider the functions

$$f_m : \mathbb{R}^n \to \mathbb{R}, f(x) = \|x\|^m,$$

for integers $m \geq 1$.

(a) Show that f_m is of class C^∞ for even m.

(b) Show that f_1 is differentiable on $\mathbb{R}^n \setminus \{0\}$, but not at zero.

(c) Is f_3 of class C^1?

Problem 3 [**6 points**]
Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be of class C^1.

(a) Is the following generalization of the mean-value theorem true of false (prove your answer)? For every $x, y \in \mathbb{R}^m$ there is a $z \in \mathbb{R}^m$ such that

$$f(x) - f(y) = Df(z)(x - y).$$

(b) Show that for all $x, y \in \mathbb{R}^m$ we have

$$\|f(x) - f(y)\| \leq C\|x - y\|,$$

with $C = \sup_{z\in I_{xy}} \|Df(z)\|$, where I_{xy} is the line segment connecting x and y, and where $\|A\| := \sup_{\|x\|=1} \|Ax\|$ is the operator norm of a linear operator A.

(c) Using (b), show that for all $x, y, a \in \mathbb{R}^m$ we have

$$\|f(x) - f(y) - Df(a)(x - y)\| \leq \|x - y\| \sup_{z\in I_{xy}} \|Df(z) - Df(a)\|.$$

Problem 4 [4 points]

(a) For each $k \geq 1$, give examples of functions $f_k : \mathbb{R} \to \mathbb{R}$ which are C^k but not C^{k+1}.

(b) Give an example of a bijective function $g : \mathbb{R} \to \mathbb{R}$ which is C^∞ but whose inverse is not C^∞.