Jacobs University
Fall 2019

October 3, 2019

Calculus on Manifolds

Homework 4

Due on October 10, 2019

Problem 1 [4 points]

Do Problem 2 (b) and the smoothness part of Problem 3 from the previous homework sheet again, with the definition of smoothness from class.

Problem 2 [4 points]

Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a bijective C^1 function. Prove that then $m \leq n$. (Remark: In class, we used this to prove that the dimension of a manifold is diffeomorphism invariant.)

Problem 3 [4 points]

Consider the n-sphere $\mathbb{S}^n$ as a manifold with the atlas defined by the stereographic projections. This atlas is smooth as we showed in the last homework sheet. Prove that the antipodal map $\alpha : \mathbb{S}^n \to \mathbb{S}^n, \alpha(x) = -x$ is smooth.

Problem 4 [8 points]

For any topological space M, let $C(M)$ denote the set of continuous functions $f : M \to \mathbb{R}$. This set $C(M)$ is an algebra over $\mathbb{R}$, meaning that the operation of multiplication (defined pointwise) is distributive and bilinear. For any continuous map $F : M \to N$ (N another topological space), we define $F^* : C(N) \to C(M), F^*(f) = f \circ F$.

(a) Show that F^* is a linear map.

(b) Now let M and N be smooth manifolds. Show that F is smooth if and only if $F^*(C^\infty(N)) \subset C^\infty(M)$.

(c) Let $F : M \to N$ be a homeomorphism between the smooth manifolds M and N. Show that F is a diffeomorphism if and only if F^* is a bijective linear map (i.e., an isomorphism) from $C^\infty(N)$ to $C^\infty(M)$.