Jacobs University — October 10, 2019
Fall 2019

Calculus on Manifolds

Homework 5

Due on October 22, 2019

Problem 1 [**4 points**]

(a) Let $U \subset \mathbb{R}^n$ be a convex neighborhood of the origin and $f : U \to \mathbb{R}$ be smooth. Prove that there exist smooth functions $g_1, \dots, g_n : U \to \mathbb{R}$ such that

$$f(x) = \sum_{j=1}^{n} x_j g_j(x) \quad \text{and} \quad g_j(0) = \frac{\partial f(0)}{\partial x^j} \; \forall\, j = 1, \dots, n.$$

(b) Let $U \subset \mathbb{R}^n$ be an open ball, $a \in U$ and $\omega : C^\infty(U) \to \mathbb{R}$ a derivation at a. Prove that there exists a unique vector $v \in \mathbb{R}^n$ such that $\omega(f) = D_v|_a f$ (the directional derivative of f at a in direction v).

Problem 2 [**4 points**]

Let M be a smooth manifold, $p \in M$, and let $\mathcal{V}_pM$ denote the set of equivalence classes of smooth curves $\gamma : (-1, 1) \to M$ with $\gamma(0) = p$ under the equivalence relation $\gamma_1 \sim \gamma_2$ if $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ for every smooth real-valued function f defined in a neighborhood of p. Show that the map $\Psi : \mathcal{V}_pM \to T_pM$ defined by $\Psi[\gamma] = d\gamma_0(\partial)$ is well-defined and bijective. Here ∂ is the usual derivative in $\mathbb{R}$, and $d\gamma_0$ is the differential of the map γ as defined in class (think carefully about what $d\gamma_0(\partial)$ means then). (Note: This gives us another way to think about the tangent space.)

Problem 3 [**8 points**]

Let M, N be smooth manifolds, M connected, and $F : M \to N$ smooth such that $dF_p : T_pM \to T_{F(p)}N$ is the zero map for all $p \in M$. Prove that then F is a constant map. (*Hint: Use local coordinates and reduce the question to a problem in* $\mathbb{R}^n$.)

Problem 4 [**8 points**]

Let M, N, P be smooth manifolds, $F : M \to N$ and $G : N \to P$ smooth maps, and let $p \in M$. Prove the chain rule for $d(G \circ F)_p : T_pM \to T_{G(F(p))}P$, i.e.,

$$d(G \circ F)_p = dG_{F(p)} \circ dF_p.$$

Problem 5 [8 points]

(a) Let M, N be smooth manifolds, and $F : M \to N$ a smooth map. Let (U, φ) be a smooth coordinate chart for M containing p, and (V, ψ) a smooth coordinate chart for N containing $F(p)$. Using the definition of the differential, carefully compute

$$dF_p \left(\frac{\partial}{\partial x^i}\bigg|_p \right)$$

in the local coordinates. (In each step of the computation, write down explicitly what properties you use.)

(b) Suppose two smooth charts (U_1, φ_1) and (U_2, φ_2) are given on a smooth manifold M, and $p \in U_1 \cap U_2$. Consider the transition map and use the definition of the differential to compute $\frac{\partial}{\partial x^i}\Big|_p$ in terms of the $\frac{\partial}{\partial \widetilde{x}^i}\Big|_p$, where (x^i) are the coordinate functions of φ_1 and $(\widetilde{x}^i)$ the coordinate functions of φ_2.

(c) Polar coordinates in $\mathbb{R}^2$ are given by the coordinate change $(x, y) = (r\cos\theta, r\sin\theta)$. Compute $\frac{\partial}{\partial r}\Big|_p$ and $\frac{\partial}{\partial \theta}\Big|_p$ in terms of $\frac{\partial}{\partial x}\Big|_p$ and $\frac{\partial}{\partial y}\Big|_p$ for any $p \neq 0$.