Jacobs University
Fall 2019

October 31, 2019

Calculus on Manifolds

Homework 6

Due on November 7, 2019

Problem 1 [**4 points**]

(a) Show that the solution set of the equation

$$x^3 + y^3 + z^3 = 1$$

is a submanifold of dimension two in $\mathbb{R}^3$.

(b) Define the function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = x^3 - 6xy + y^2.$$

Find all values of $c \in \mathbb{R}$ for which the level set $f^{-1}(c)$ is a submanifold of $\mathbb{R}^2$.

Problem 2 [**3 points**]

Let $f : \mathbb{R}^n \to \mathbb{R}$ be smooth and a a regular value of f. Then $M = f^{-1}(a)$ is a submanifold of $\mathbb{R}^n$. Show that for every $p \in M$, the tangent space T_pM viewed as a subspace of $\mathbb{R}^n$ is given by

$$T_pM = \left\{ (v_1, \dots, v_n) \in \mathbb{R}^n : \sum_{i=1}^{n} v_i \frac{\partial f}{\partial x^i}(p) = 0 \right\}.$$

Problem 3 [**6 points**]

Let $\det : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}$ denote the determinant function.

(a) Using matrix entries (X_i^j) as global coordinates on $\mathrm{GL}(n, \mathbb{R})$, show that the partial derivatives of det are given by

$$\frac{\partial}{\partial X_i^j}(\det X) = (\det X)(X^{-1})_j^i.$$

(b) Conclude that the differential of the determinant function is

$$d(\det)_X(B) = (\det X)\mathrm{tr}(X^{-1}B),$$

for $X \in \mathrm{GL}(n, \mathbb{R})$ and $B \in T_X\mathrm{GL}(n, \mathbb{R}) \simeq M(n, \mathbb{R})$, where $\mathrm{tr}X = \sum_{i=1}^{n} X_i^i$ is the trace of X.

(c) Show that $\det : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}$ is a smooth submersion.

Problem 4 [3 points]

In class we discussed the following lemma: Let M be a compact m-dimensional smooth manifold, $F : M \to \mathbb{R}^N$ a smooth injective immersion, and $N > 2m + 1$. Then there is a dense set of vectors $v \in \mathbb{S}^{N-1}$ such that $\pi_v \circ F$ is a smooth injective immersion, where $\pi_v(x) = x - \langle x, v \rangle v$ is the projection orthogonal to $v \in \mathbb{S}^{N-1}$. Finish the proof from class by showing that $\pi_v \circ F$ is indeed an immersion.

Problem 5 [4 points]

Finish the proof of the Whitney Embedding Theorem from class by showing that the map F that was defined is indeed injective and an immersion.