Jacobs University November 14, 2019
Fall 2019

Calculus on Manifolds

Homework 8

Due on November 21, 2019

Problem 1 [**3 points**]
Let X and Y be two vector fields on a smooth manifold M. Prove that the Lie bracket $[X,Y] : C^\infty(M) \to C^\infty(M), [X,Y]f = XYf - YXf$ is a global derivation. Is the map $D_{XY} : C^\infty(M) \to C^\infty(M), D_{XY}(f) = XYf$ also a derivation? (Prove your answer or give a counter example.)

Problem 2 [**4 points**]

(a) Prove the local coordinate formula for Lie brackets that was stated in class, i.e.,

$$[X,Y] = \sum_{i,j=1}^{n} \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}.$$

(b) Compute the Lie bracket explicitly for the following vector fields in $\mathbb{R}^3$:

$$X = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + yz\frac{\partial}{\partial z}$$
$$Y = y\frac{\partial}{\partial x} + x(z+1)\frac{\partial}{\partial z}.$$

Problem 3 [**5 points**]
Prove the properties a) - e) of the Lie bracket that we discussed in class, i.e., bilinearity, antisymmetry, the Jacobi identity, behavior under multiplication with C^∞ functions, and behavior under pushforwards.

Problem 4 [**2 points**]
Consider the vector field $X = x^2 \frac{\partial}{\partial x}$ on $\mathbb{R}^2$. Find the integral curves. Is the flow generated by X global?

Problem 5 [**4 points**]
Let M be the open submanifold of $\mathbb{R}^2$ where both x and y are positive, and let $F : M \to M$ be the map $F(x,y) = (xy, \frac{y}{x})$. Show that F is a diffeomorphism, and compute F_*X and F_*Y, where

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad Y = y\frac{\partial}{\partial x}.$$

Problem 6 [**2 points**]
Compute the flow of the vector field $X = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}$ on $\mathbb{R}^2$.