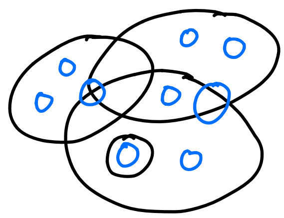
Session 4
Sep. 12, 2019

Thm.: Let (X,τ) be second-countable. Then every open cover of X has a countable subcover ($=$ Lindelöf space).

Proof: Idea: we index some sets of the open cover by basis elements s.t. we still have a subcover

- $\mathcal{B}$ countable basis
- $\mathcal{U}$ some open cover
- $\mathcal{B}_{\mathcal{U}} = \{B \in \mathcal{B} : B \subset V \text{ for some } V \in \mathcal{U}\}$
- for each $B \in \mathcal{B}_{\mathcal{U}}$ choose one $V_B \in \mathcal{U}$, s.t. $B \subset V_B$



$\Rightarrow$ $\mathcal{U}_c = \{V_B : B \in \mathcal{B}_{\mathcal{U}}\}$ is countable; does it still cover X?

pick some $y \in X$ (to show: $y \in V_B$ for some $V_B \in \mathcal{U}_c$)

↳ there is $V \in \mathcal{U}$, $y \in V$ ($\mathcal{U}$ open cover)

↳ there is $B \in \mathcal{B}$, $y \in B \subset V$ ($\mathcal{B}$ basis)

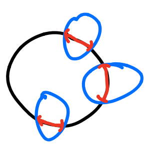
$\Rightarrow B \in \mathcal{B}_{\mathcal{U}} \Rightarrow y \in B \subset V_B$ for some $V_B \in \mathcal{U}_c \Rightarrow \mathcal{U}_c$ open cover $\square$

next: subspaces and products

Def.: (X,τ) top. space, $S \subset X$. Then the subspace topology on S is

$$\tau_S = \{U \subset S : U = V \cap S \text{ for some } V \in \tau\}$$

Ex.: natural top. on a circle



Def.: Let $(X_1,\tau_1),\dots,(X_k,\tau_k)$ be top. spaces. The product topology on $X_1 \times X_2 \times \dots X_k$ (Cartesian product) is the top. generated by $\{U_1 \times \dots \times U_k : U_i \in \tau_i, i=1,\dots,k\}$, the corresponding top. space is called product space.

<u>Def.:</u> $\pi_i : X_1 \times \dots \times X_k \to X_i$, $\pi_i(x_1,\dots,x_k) = x_i$ is called i-th canonical projection.

note: • π_i's are continuous: $\pi_i^{-1}(U_i) = X_1 \times \dots \times U_i \times \dots \times X_k$

• $f: Y \to X_1 \times \dots \times X_k$ cont. $\Longleftrightarrow$ $f_i = \pi_i \circ f : Y \to X_i$ (component fct.s) cont.

("$\Rightarrow$" composition of cont. fct.s ; "$\Leftarrow$" by def.)

next: compactness: "finiteness conclusions on infinite sets"

<u>Def.:</u> A top. space (X,τ) is compact if every open cover of X has a finite subcover.

note: compact subset means it is compact in subspace topology

recall main results from Analysis:

- If (X,τ) compact, then every continuous $f: X \to \mathbb{R}$ assumes its maximum and minimum.
- Heine-Borel: $X \subset \mathbb{R}$ compact $\Leftrightarrow$ X closed and bounded
- $f: X \to Y$ cont., X compact $\Rightarrow$ $f(X)$ compact
- $f: M_1 \to M_2$, (M_1,d_1) and (M_2,d_2) metric spaces, $C \subset M_1$ compact. Then f cont. $\Rightarrow$ $f|_K$ uniformly cont.

note: X second-countable Hausdorff or metric space:

X compact $\Leftrightarrow$ every sequence in X has a convergent subsequence with limit in X
(= sequential compactness)

next: (path-) connectedness

<u>Def.:</u> A top. space (X,τ) is connected if the only subsets of X that are both open and closed are X and $\emptyset$.

note: (X,τ) disconnected $\Longleftrightarrow \exists U,V$ non-empty, disjoint and open s.t. $X = U \cup V$

("$\Rightarrow$" $\exists U$ open and closed, $U \neq X, U \neq \emptyset \Rightarrow U^c$ open and closed $\Rightarrow U \cup U^c = X$
"$\Leftarrow$" $U^c = V$ open and closed and neither $= \emptyset$ nor $= X$)

Ex.: $[0,1] \cup [2,3]$ with subspace topology

$\hookrightarrow \underbrace{B_1(\tfrac{1}{2})}_{=(-\frac{1}{2},\frac{3}{2})} \cap ([0,1] \cup [2,3]) = [0,1]$ is open, but also closed $\Rightarrow$ disconnected

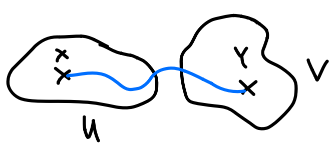
note: • for $S \subset X$, the boundary of S is def. as $\partial S = \{p \in X :$ all neighborhoods of p have at least one point in S and one not in $S\}$

• S both open and closed $\Longleftrightarrow \partial S = \emptyset$

<u>Def.:</u> A top. space (X,τ) is path-connected if $\forall x,y \in X \ \exists$ cont. $\gamma : [0,1] \to X$ with $\gamma(0) = x, \gamma(1) = y$ (γ = path)

note: • path conn. $\Longrightarrow$ conn.

(proof:



path has to cross boundary, but sets that are both open and closed have no boundary)

• for open subsets of $\mathbb{R}^n$: path conn. $\Longleftrightarrow$ conn.