2.5 Tangent Space

Session 9
Oct. 8, 2019

A few words about Linear Algebra:

Def.: A vector space over $\mathbb{R}$ (or any field $\mathbb{F}$) is a set V with two operations:

- addition: $V \times V \to V$
- scalar multiplication: $\mathbb{R} \times V \to V$

that satisfy:

- V is an abelian group under addition, i.e., associative, commutative, $\exists$ zero, $\exists$ inverse
- $\lambda_1(\lambda_2 v) = (\lambda_1 \lambda_2) v \quad \forall v \in V, \lambda_1, \lambda_2 \in \mathbb{R}$

  $1 \cdot v = v \quad \forall v \in V$
- distributive: $(\lambda_1 + \lambda_2) v = \lambda_1 v + \lambda_2 v \quad \forall v \in V, \lambda_1, \lambda_2 \in \mathbb{R}$

  $\lambda(v+w) = \lambda v + \lambda w \quad \forall v, w \in V, \lambda \in \mathbb{R}$

Ex.s: $\mathbb{R}^n$, $n \times m$ matrices, Polynomials of degree $\leq n$

Recall notions of linear combination, linear (in)dependence, basis

Def.: A map $T: V \to W$, V, W vector spaces is linear if $T(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 T(v_1) + \lambda_2 T(v_2)$

$\forall v_1, v_2 \in V, \lambda_1, \lambda_2 \in \mathbb{R}$

A bijective linear map is called isomorphism.

Note:
- f isomorphism $\Rightarrow f^{-1}$ linear
- V, W isomorphic ($\exists$ isomorphism) $\Leftrightarrow \dim V = \dim W$
- canonical isomorphisms are those that involve no arbitrary choices (e.g., of basis)
- all norms on finite-dim. vector spaces are equivalent ($\|v\|_1 \leq c \|v\|_2 \leq \tilde{c} \|v\|_1$)

  $\Rightarrow$ all lead to same topology

  given basis $(e_1, \dots, e_n)$, def. isomorphism $E: \mathbb{R}^n \to V$, $E(x) = \sum_{i=1}^n x^i e_i$ $((V, E^{-1})$ chart$)$

  $\Rightarrow$ topological manifold

$(\tilde{e}_1, \dots, \tilde{e}_n)$ another basis, $\tilde{E}(x) = \sum_{i=1}^{n} x^i \tilde{e}_i$, $\exists$ matrix of basis change A_i^j, i.e., $e_i = \sum_{j=1}^{n} A_i^j \tilde{e}_j$

$\Rightarrow$ transition map $\tilde{E}^{-1} \circ E(x) = \tilde{E}^{-1}\left(\sum_{i=1}^{n} x^i e_i\right) = \sum_{i=1}^{n} x^i \underbrace{\tilde{E}^{-1}(e_i)}_{= \tilde{E}^{-1}\left(\sum_{j=1}^{n} A_i^j \tilde{e}_j\right)} = \left(\sum_i A_i^1 x^i, \dots, \sum_i A_i^n x^i\right)$

↳ diffeomorphism

$\Rightarrow \{(V, E^{-1}) : E$ def. via basis$\}$ smooth atlas $\Rightarrow$ smooth manifold

$\Rightarrow$ standard smooth structure

Next: def. derivatives for $f: M \to N$

Ideas:
- need linear approximation to f
-  derivative lives in what we will call the tangent space
  ↳ but: want def. independent of any embedding in higher-dim. space
- also want def. indep. of coordinates
- rough idea: space of all directional derivatives

Recall: directional derivative of $f: \mathbb{R}^n \to \mathbb{R}$ at point a in direction v:

$$D_v|_a f = D_v f(a) = \frac{d}{dt} f(a+tv)\Big|_{t=0} = \sum_i v^i \frac{\partial f}{\partial x^i}(a)$$

↳ product rule: $D_v(fg)(a) = f(a) D_v g(a) + g(a) D_v f(a)$

Def.: For $a \in \mathbb{R}^n$, a map $w: C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is called derivation at a if it is linear and satisfies Leibniz rule $w(fg) = f(a)\, wg + g(a)\, wf \quad \forall f, g \in C^\infty(\mathbb{R}^n)$.

$T_a\mathbb{R}^n := \{w : w$ derivation at $a\}$

Note:
- $D_v|_a$ is a derivation
- some computations show that $T_a\mathbb{R}^n$ is isomorphic to $\mathbb{R}^n$ (or $\mathbb{R}^n_a = \{a\} \times \mathbb{R}^n$, the geometric tangent space)
  ↳ $v \mapsto D_v|_a$ is isomorphism (see next HW)
- $\left(\frac{\partial}{\partial x^1}\Big|_a, \dots, \frac{\partial}{\partial x^n}\Big|_a\right)$ is a basis of $T_a\mathbb{R}^n$ $\left(\frac{\partial}{\partial x^i}\Big|_a f := \frac{\partial f}{\partial x^i}(a)\right)$