Session 11
Oct. 10, 2019

3. Embeddings, Submanifolds, Sard's Theorem

3.1 Local Structure of Maps between Manifolds

recall from linear Algebra:

- linear map $T: V \to W$, $\operatorname{im}(T) = \{Tv \in W : v \in V\}$, $\operatorname{rank}(T) = \dim(\operatorname{im} T)$
  $\ker(T) = \{v \in V : Tv = 0\}$, $\operatorname{nullity}(T) = \dim(\ker T)$
- for any linear map T of rank r one can choose bases s.t. matrix of $T = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow \dim V = \dim(\operatorname{im} T) + \dim(\ker T)$

$\Rightarrow$ up to basis choice, a linear map is only characterized by its rank

- T injective $\Leftrightarrow \ker T = \{0\} \Leftrightarrow \dim(\operatorname{im} T) = \dim V \Leftrightarrow T = \begin{pmatrix} I_r \\ 0 \end{pmatrix}$ in some basis
- T surjective $\Leftrightarrow \dim(\operatorname{im} T) = \dim W \Leftrightarrow T = (I_r, 0)$ in some basis

back to manifolds:

Def.: let M, N be smooth manifolds, $F: M \to N$ smooth. Then we call F

- submersion if dF_p is surjective $\forall p \in M$ ($\operatorname{rank} dF_p = \dim N$)
- immersion if dF_p is injective $\forall p \in M$ ($\operatorname{rank} dF_p = \dim M$)
- embedding if F is an immersion and $F: M \to F(M)$ a homeomorphism.

If $\operatorname{rank} dF_p = r$ $\forall p \in M$, we say F has constant rank ($\operatorname{rank} F = r$).

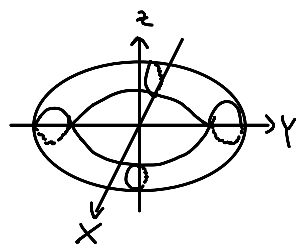
Ex.s: • projections $\pi_i: M_1 \times \ldots \times M_k \to M_i$ are submersions

• smooth curves $\gamma: [0,1] \to M$ with $\gamma'(t) \neq 0 \; \forall t \in [0,1]$ are immersions

(e.g., $\gamma(t) = (t^3, 0)$ is not an immersion since $\gamma'(0) = (0,0)$)

• $U \subset M$ open, inclusion map $i: U \to M$ is an embedding

• $F: \mathbb{R}^2 \to \mathbb{R}^3$, $F(u,v) = \big((2+\cos 2\pi u)\cos(2\pi v), (2+\cos 2\pi u)\sin(2\pi v), \sin(2\pi u) \big)$ is an immersion



with some work we can show that $F: S^1 \times S^1 \to \mathbb{R}^3$ is also an embedding

next: consider such submanifolds (like $\mathrm{Im}\, F$, which is not open in $\mathbb{R}^3$)

Important result: Rank-thm.: If $F: M \to N$ has constant rank r, then locally

$$\hat{F}(x^1, \ldots, x^r, x^{r+1}, \ldots, x^m) = (x^1, \ldots, x^r, 0, \ldots, 0).$$

$\Rightarrow F$ submersion: $\hat{F}(x^1, \ldots, x^m) = (x^1, \ldots, x^n)$ $\quad (m > n)$

$\Rightarrow F$ immersion: $\hat{F}(x^1, \ldots, x^m) = (x^1, \ldots, x^m, 0, \ldots, 0)$ $\quad (m < n)$.