3.4 Whitney Embedding Theorem

Session 15
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Result: embedding of a smooth m-dim. manifold M into $\mathbb{R}^{2m+1}$

two steps: 1) if embedding into $\mathbb{R}^N$ for $N > 2m+1$, then $\exists$ embedding into $\mathbb{R}^{N-1}$

2) embedding into some $\mathbb{R}^N$

Here: consider only M compact, since then:

Recall: M compact, $F: M \to N$ smooth injective $\underbrace{\text{immersion}}_{dF_p \text{ injective } \forall p \in M}$ $\Rightarrow$ F embedding

We use Sard's thm. to show:

<u>Corollary</u>: Let $F: M \to N$ be smooth. Then the set of regular values of F is dense.

(Dense = interior of complement empty).

Proof: Cover M with countably many open U_i, choose open V_i s.t. $\overline{V_i} \subset U_i$ and $\{V_i\}$ still a cover. Set C_F = critical points of F.

Then, write $F(C_F) = F\left(\bigcup_{i=1}^{\infty} \overline{V_i} \cap C_F\right) = \bigcup_{i=1}^{\infty} \underbrace{F(\overline{V_i} \cap C_F)}_{\substack{\text{compact set of measure zero (Sard)} \\ \Rightarrow \text{empty interior}}}$

$\Rightarrow F(C_F)$ has empty interior $\Rightarrow F(R_F)$ dense (R_F = regular points) $\square$

this implies:

<u>Lemma</u>: Let M be a compact smooth m-dim. manifold, N smooth n-dim. manifold, $m < n$, and $F: M \to N$ smooth. Then $N \setminus f(M)$ is a dense open subset of N.

Proof: $m<n$, so every $p\in M$ is a critical point $\Rightarrow F(M)$ has measure zero

by previous corollary regular values $R_F = N\setminus F(M)$ dense

M compact $\Rightarrow F(M)$ closed $\Rightarrow N\setminus F(M)$ open. $\square$

Def.: Let $v\in S^{N-1}\subset\mathbb{R}^N$, then we def. the orthogonal projection

$$\pi_v(x) = x - \langle x,v\rangle v \quad \forall x\in\mathbb{R}^N.$$

Note: $\pi_v(v) = v - \underbrace{\langle v,v\rangle}_{=1\ (v\in S^{N-1})} v = 0$

Lemma: Let M be a compact m-dim. smooth manifold, $F: M\to\mathbb{R}^N$ a smooth injective immersion, and $N > 2m+1$. Then there is a dense set of vectors $v\in S^{N-1}$ s.t. $\pi_v\circ F$ is a smooth inj. immersion $M\to\mathbb{R}^{N-1}$.

Proof: set $X = F(M)\subset\mathbb{R}^N$

- $\pi_v\circ F$ injective: need $\pi_v(x)\neq\pi_v(y)$ $\forall x,y\in X$, i.e.,

$x - \langle x,v\rangle v \neq y - \langle y,v\rangle v \iff \frac{x-y}{\|x-y\|}\neq v$

now def. $\Delta_X = \{(x,x): x\in X\}$, the diagonal of $X\times X$, and def.

$h: \underbrace{X\times X\setminus\Delta_X}_{\dim = 2m} \to \underbrace{S^{N-1}}_{\dim = N-1}$, $h(x,y) = \frac{x-y}{\|x-y\|}$

need $v\notin h(X\times X\setminus\Delta_X)$

by previous Lemma a dense set of such v exists as long as $2m < N-1$.

- $\pi_v \circ F$ immersion: HW

Theorem (Whitney embedding compact case):

Any compact smooth m-dim. manifold M can be embedded into $\mathbb{R}^{2m+1}$.

Proof: show embedding into some $\mathbb{R}^d$, then $d = 2m+1$ follows from previous lemma

M compact $\Rightarrow$ can choose finite open cover $\{U_i\}_{i=1,\dots,N}$, corresponding charts (U_i, φ_i).

choose new open cover $\{V_i\}$ s.t. $\overline{V_i} \subset U_i$.

def. bump fct.s $\rho_i : M \to \mathbb{R}$, s.t. $\rho_i|_{\overline{V_i}} = 1$ and $\operatorname{supp} \rho_i \subset U_i$

def. $F := (\rho_1 \varphi_1, \dots, \rho_m \varphi_m, \rho_1, \dots, \rho_m)$ $\quad \left(\rho_i \varphi_i|_{(\operatorname{supp} \rho_i)^c} := 0\right)$

HW: show that this F is injective and an immersion.

Since M compact, F is an embedding. $\square$