Session 21
Nov. 21, 2019

<u>Def.</u>: The cotangent space at $p \in M$ is $T_p^*M := (T_pM)^*$, $\omega \in T_p^*M$ is called (tangent) covector at p.

Notes: in coordinates, $\left\{\frac{\partial}{\partial x^i}\Big|_p\right\}$ basis of $T_pM \Rightarrow$ dual basis $\{\lambda^i|_p\}$ of T_p^*M

$$\Rightarrow T_p^*M \ni \omega = \omega_i \lambda^i|_p \quad \text{with } \omega_i = \omega\left(\frac{\partial}{\partial x^i}\Big|_p\right)$$

<u>Def.</u>: • $T^*M = \bigsqcup_{p \in M} T_p^*M$ is called cotangent bundle of M.

• $\omega: M \to T^*M$ with $\omega(p) \in T_p^*M$ is called covector field (= differential 1-form)

Note: one can show that T^*M is a smooth manifold

Note: ω covector field, X vector field

$$\Rightarrow \omega(X): M \to \mathbb{R}, \quad \omega(X)(p) = \omega(p)X(p)$$

in local coordinates: $\omega(X) = \omega_i X^i$

Now: for $f \in C^\infty(M)$, the differential at p was def. as $df_p: T_pM \to T_{f(p)}\mathbb{R}$

We can also regard the differential of f as the covector field df, def. by $df_p(v) = vf$ $\forall v \in T_pM$

i.e., $T_p^*M \ni df_p: T_pM \to \mathbb{R}$ $\Rightarrow$ same map with the identification of $T_{f(p)}\mathbb{R}$ with $\mathbb{R}$

In coordinates: $T_p^*M \ni df_p = A_i(p)\lambda^i|_p$ for some smooth A_i

$$\Rightarrow A_i(p) = df_p\left(\frac{\partial}{\partial x^i}\Big|_p\right) := \frac{\partial}{\partial x^i}\Big|_p f = \frac{\partial f}{\partial x^i}(p)$$

$$\Rightarrow df_p = \frac{\partial f}{\partial x^i}(p)\,\lambda^i|_p$$

for $f = x^j$ we find $dx^j|_p = \lambda^j|_p \Rightarrow \lambda^j = dx^j \Rightarrow df = \frac{\partial f}{\partial x^i}dx^i$

now, consider smooth $F: M \to N$

recall that for diffeomorphisms F and $X \in \mathfrak{X}(M)$ we defined the pushforward

$$F_*X: N \to TN, \; F_*X(q) = dF_{F^{-1}(q)}(X(F^{-1}(q))) \qquad (F_*X \text{ is a new vector field})$$

now take any smooth $F: M \to N$ (not necessarily a diffeomorphism)

<u>Def.:</u> $F: M \to N$ smooth, $\omega: N \to T^*N$ a covector field, then the pullback of ω by F is def. as

$$F^*\omega: M \to T^*M, \; (F^*\omega)_p = dF_p^*(\omega_{F(p)}).$$

note: one can prove that $F^*\omega$ is a smooth covector field

5.4 Tensors

<u>Def.:</u> • $\underbrace{A: V_1 \times \dots \times V_k \to W}_{\text{vector spaces}}$ is called multilinear if

$$A(v_1, \dots, \lambda_i v_i + \tilde{v}_i, \dots, v_k) = \lambda_i A(v_1, \dots, v_k) + A(v_1, \dots, \tilde{v}_i, \dots, v_k)$$

• $L(V_1, \dots, V_k; W) =$ set of all multilinear maps $V_1 \times \dots \times V_k \to W$

• If $A: V_1 \times \dots \times V_k \to \mathbb{R}$ and $B: W_1 \times \dots \times W_\ell \to \mathbb{R}$ are multilinear then

$A \otimes B: V_1 \times \dots \times V_k \times W_1 \times \dots \times W_\ell \to \mathbb{R}$, $A \otimes B(v_1, \dots, v_k, w_1, \dots, w_\ell) = A(v_1, \dots, v_k) B(w_1, \dots, w_\ell)$

is called tensor product of A and B.

Ex.: $\omega^j \in V_j^*$, then $\omega^1 \otimes \dots \otimes \omega^n: V_1 \times \dots \times V_k \to \mathbb{R}$, $(v_1, \dots, v_k) \mapsto \omega^1(v_1) \dots \omega^k(v_k)$

Note: • $\mathcal{B} = \{\varepsilon_{(1)}^{i_1} \otimes \dots \otimes \varepsilon_{(k)}^{i_k} : 1 \le i_1 \le n, \dots, 1 \le i_k \le n\}$ basis of $L(V_1, \dots, V_k; \mathbb{R})$

• more abstractly one can define a space $V_1 \otimes \dots \otimes V_k$; here, just take it as the vector space with basis $\mathcal{C} = \{E_{(1)}^{i_1} \otimes \dots \otimes E_{(k)}^{i_k} : 1 \le i_1 \le n, \dots, 1 \le i_k \le n\}$