Session 5
Sep. 16, 2019

Immunization:

$\rightarrow$ look for $\dfrac{\partial FV_m(r)}{\partial r} = 0$

$$\Rightarrow F_1 (m-n_1)(1+r)^{m-n_1-1} + F_2 (m-n_2)(1+r)^{m-n_2-1} = 0$$

$$\Rightarrow F_1 (m-n_1)(1+r)^{-n_1} + F_2 (m-n_2)(1+r)^{-n_2} = 0$$

$$\Rightarrow m \underbrace{\frac{F_1}{(1+r)^{n_1}}}_{P_1} + m \underbrace{\frac{F_2}{(1+r)^{n_2}}}_{P_2} = n_1 \underbrace{\frac{F_1}{(1+r)^{n_1}}}_{P_1} + n_2 \underbrace{\frac{F_2}{(1+r)^{n_2}}}_{P_2}$$

?

$$\Rightarrow \text{def. } P := P_1 + P_2 \Rightarrow mP = n_1 P_1 + n_2 P_2$$

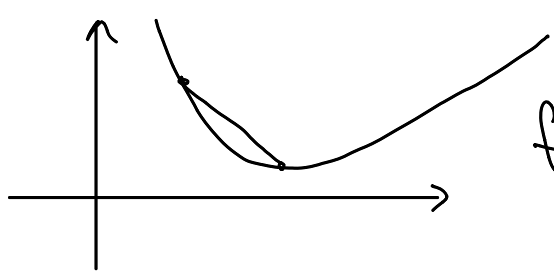
$$\Rightarrow m = \underbrace{\frac{1}{P}(n_1 P_1 + n_2 P_2)}_{=: MD = \text{Macaulay duration}} \rightarrow \text{weighted average}$$

note: $m = (1+r)\underbrace{\left(-\frac{1}{P}\frac{\partial P}{\partial r}\right)}_{\text{generally called price volatility}}$ $\qquad P = \dfrac{F_1}{(1+r)^{n_1}} + \dfrac{F_2}{(1+r)^{n_2}}$

gen. strategy: choose portfolio with $MD = m$

to have a minimum, we need $FV(r)$ to be convex (for a certain range of r)

Convex:



$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$
$$\forall \lambda \in [0,1]$$

in our case:

$$\frac{\partial^2 FV(r)}{\partial r^2} = F_1 \underbrace{(m-n_1)(m-n_1-1)}_{\geq 0}(1+r)^{m-n_1-2} + F_2 \underbrace{(m-n_2)(m-n_2-1)}_{\geq 0}(1+r)^{m-n_2-2}$$

$\Rightarrow$ minimum

To summarize, the general immunization conditions are:

(1) $FV = L$ at horizon m

(2) $\frac{\partial FV(r)}{\partial r} = 0$, or $MD = m$

(3) $FV(r)$ convex around relevant r

a few remarks:

- gen. cash flows $\sum_{i=1}^{n} \frac{C_i}{(1+r)^i}$

Macaulay duration $MD = \frac{1}{P} \sum_{i=1}^{n} \frac{i C_i}{(1+r)^i} = (1+r)\left(-\frac{1}{P}\frac{\partial P}{\partial r}\right)$

- for level-coupon bond: $P = \sum_{i=1}^{n} \frac{C}{(1+r)^i} + \frac{F}{(1+r)^n}$, $C = c \cdot F$
(m=1)

volatility $-\frac{1}{P}\left(\frac{\partial P}{\partial r}\right) = \dfrac{\frac{c}{r} \cdot n - \frac{c}{r^2}(1+r)\left((1+r)^n - 1\right) - n}{\frac{c}{r}(1+r)\left((1+r)^n - 1\right) + (1+r)}$

$$MD = \frac{c(1+r)\left((1+r)^n - 1\right) + nr(r-c)}{cr\left((1+r)^n - 1\right) + r^2}$$

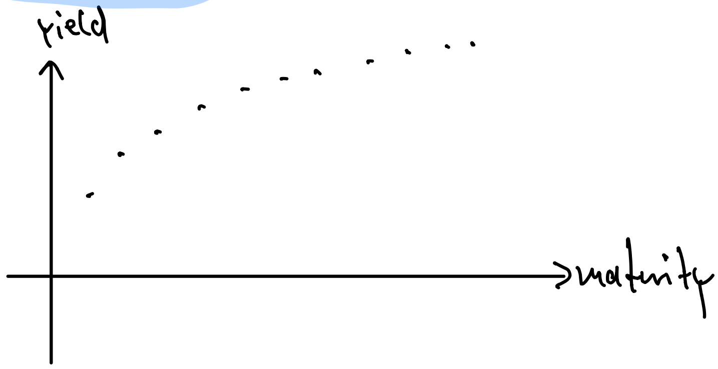
1.5 Spot Rates

yield should be different depending on maturity date

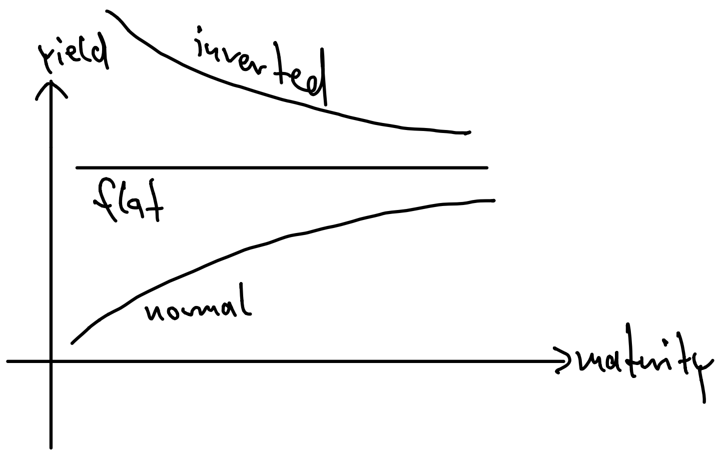
(usually: longer commitment $\Rightarrow$ more interest)

$\Rightarrow$ called "term structure"

yield curve:



types of curves:



spot rate $S(i) :=$ yield to maturity of i-period zero-coupon bond

$$P = \frac{F}{(1+S(i))^i}$$ $\Rightarrow$ spot rate curve = zero-coupon bond yield curve

better price for level-coupon bonds (given, say, "riskless" US treasury bond to determine $S(i)$):

$$P = \sum_{i=1}^{n} \frac{C}{(1+S(i))^i} + \frac{F}{(1+S(n))^n}$$, $d(i) := (1+S(i))^{-i}$ called discount factor

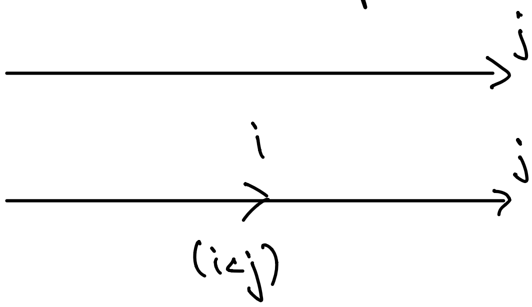
$$P = \sum_{i=1}^{n} d(i) C + d(n) F$$

called "static spread"

risky bonds should be cheaper: replace $(1+S(i))^{-i}$ by $(1+s+S(i))^{-i}$.

Forward Rates:

consider zero-coupon bonds:



$$FV_j = P(1+S(j))^j$$

$$FV_i = P(1+S(i))^i$$

$$FV_j = P(1+S(i))^i(1+S(i,j))^{j-i}$$

$S(i,j) :=$ $(j-i)$-period spot rate i periods from now (unknown)

(implied) forward rate $f(i,j)$ = guess for $S(i,j)$ based on

$$(1+S(j))^j = (1+S(i))^i(1+f(i,j))^{j-i}$$

$$\Rightarrow f(i,j) = \left(\frac{(1+S(j))^j}{(1+S(i))^i}\right)^{\frac{1}{j-i}} - 1$$