Session 7
Sep. 30, 2019

last time: binary model

$$\begin{tikzcd}[row sep=small]
 & S\cdot u \\
S \arrow[ur, "p"] \arrow[dr, "1-p"'] & \\
 & S\cdot d
\end{tikzcd}$$

recall example: $K = 3000\$$, $r = 0$, call

$$\begin{tikzcd}[row sep=small]
 & 4000\$ \arrow[r] & \text{payoff } 1000\$ \\
 & \hookrightarrow u = 1.6 & \\
S = 2500\$ \arrow[uur, "p=\frac{1}{2}"] \arrow[ddr, "p=\frac{1}{2}"'] & & \\
 & 2000\$ \arrow[r] & \text{payoff } 0\$ \\
 & \hookrightarrow d = 0.8 &
\end{tikzcd}$$

we dismissed the idea of option price = average w.r.t. stock probabilities (here: price $C = \frac{1}{2}\cdot 1000\$ + \frac{1}{2}\cdot 0\$ = 500\$$), because of the possibility of risk-free profit

new idea: option price = cost of portfolio (of bonds and stocks) that leads to no risk-free profit for seller

Such a portfolio is then called replicating portfolio.

we def.:

- x_1 = price of bond with riskless interest rate r, continuously compounded (we assume $d < e^r < u$)
- x_2 = # of stocks at price S (also called "hedge ratio" or "delta")

$\Rightarrow$ replicating portfolio costs $C = x_1 + Sx_2$

in general: riskless or zero profit for seller if two conditions hold:

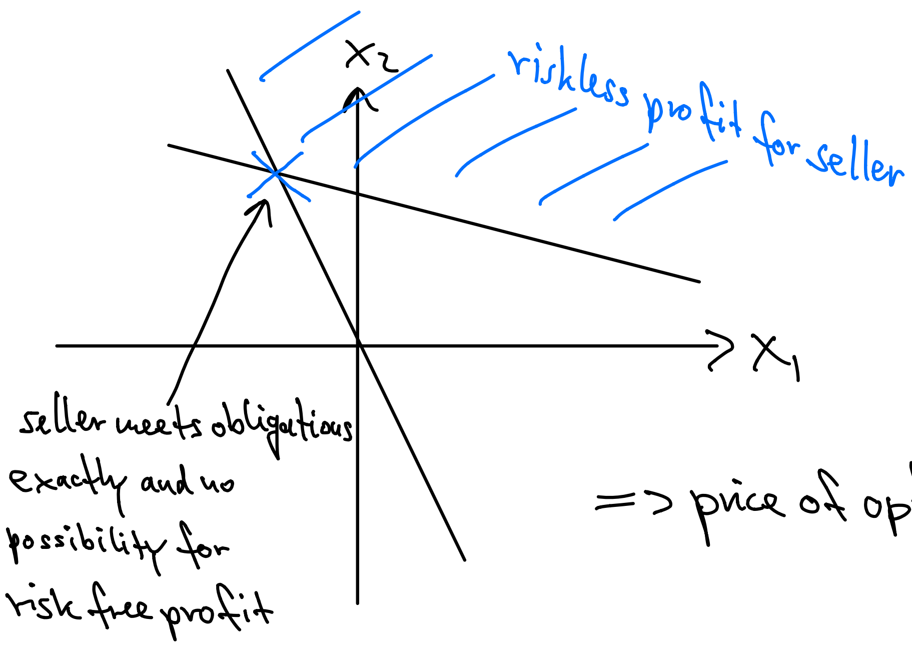
$$\underbrace{e^r x_1 + Sd\, x_2}_{\text{future value of portfolio}} \geq \underbrace{C_d}_{\text{what seller needs to pay}}$$

$$e^r x_1 + Su \cdot x_2 \geq C_u \qquad (C_u = \text{payoff at } Su)$$

$$(C_d = \text{payoff at } Sd)$$

for call options: $C_u = \max(0, Su - K)$

$C_d = \max(0, Sd - K)$



$\Rightarrow$ price of option = cost of portfolio at this intersection

$\Rightarrow$ set up replicating portfolio here

$\Rightarrow$ price $C = x_1 + S x_2$ with x_1 and x_2 determined by $e^r x_1 + Su\, x_2 = C_u$

$e^r x_1 + Sd\, x_2 = C_d$

Ex. from above: replicating portfolio $x_1 + 4000x_2 = 1000$

$$x_1 + 2000x_2 = 0$$

$\Rightarrow 2000x_2 = 1000 \Rightarrow x_2 = \frac{1}{2} \Rightarrow x_1 = -1000$

portfolio costs $-1000\,\$ + \frac{1}{2}\cdot 2500\,\$ = \boxed{250\,\$ = C}$ = option price

this is the fair price, since no possibility for risk-free profit

In detail:

	seller: borrow 1000\$, $\overbrace{\text{buy } \tfrac{1}{2} \text{ stock}}^{\text{for } 1250\,\$}$	buyer: buys option for 250 \$
$S_T = 4000\,\$$	buy $\frac{1}{2}$ stock: 2000 \$ sell stock for 3000 \$ $\Rightarrow$ profit: $-1000\,\$ - 2000\,\$ + 3000\,\$ = 0\,\$$	buy stock at $K = 3000\,\$$ $\Rightarrow$ profit: $\underbrace{1000\,\$}_{4000\,\$ - 3000\,\$} - 250\,\$ = 750\,\$$
$S_T = 2000\,\$$	sell $\frac{1}{2}$ stock: 1000 \$ $\Rightarrow$ profit: $-1000\,\$ + 1000\,\$ = 0\,\$$	do not exercise option $\Rightarrow$ profit: $-250\,\$$

note: • one can actually buy $\frac{1}{2}$ stock

↳ called "fractional share" (used, e.g., for dividend reinvestment) note: there is a put-call parity

call price C

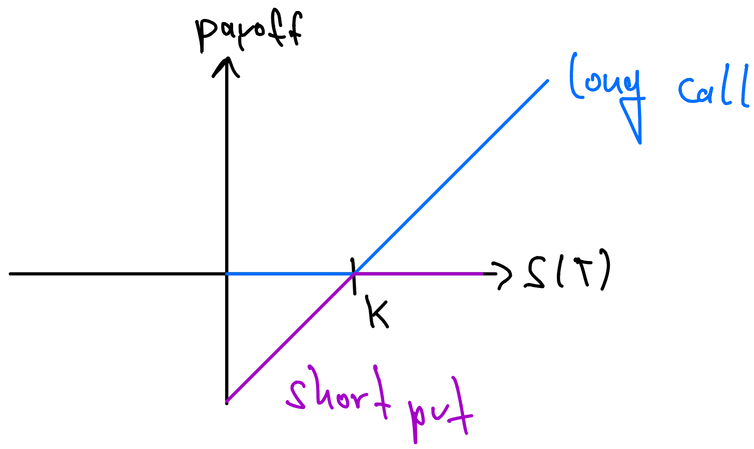
put price P

look at two possible portfolios:

- call, put with same stock, K, T

  buy call, sell put, which costs $C-P$

  $\text{payoff} = S(T) - K$



- replicating portfolio with 1 stock, borrow bonds worth K at time T

  $\Rightarrow \text{payoff} = S(T) - K$

  $\Rightarrow \text{cost} = S - e^{-rT} K$

no arbitrage $\Rightarrow$ both portfolios must have same price (since same payoff)

$$\Rightarrow C - P = S - e^{-rT} K$$

gen. solution for option price: need to solve $e^r x_1 + Su x_2 = C_u$

$$e^r x_1 + Sd x_2 = C_d$$

$$\Rightarrow Su x_2 - Sd x_2 = C_u - C_d \Rightarrow x_2 = \frac{C_u - C_d}{Su - Sd}$$

$$\Rightarrow x_1 = e^{-r}(C_d - Sd x_2)$$

$$= e^{-r}\left(C_d - \frac{Sd(C_u - C_d)}{Su - Sd}\right)$$

$$= e^{-r}\left(\frac{(u-d)C_d - d(C_u - C_d)}{u-d}\right) = e^{-r}\left(\frac{uC_d - dC_u}{u-d}\right)$$

$$\Rightarrow C = x_1 + Sx_2 = e^{-r}\left(\frac{uC_d - dC_u}{u-d}\right) + S\left(\frac{C_u - C_d}{Su - Sd}\right)$$

$$= e^{-r}\left(C_d \underbrace{\frac{u - e^r}{u-d}}_{=: p_d} + C_u \underbrace{\frac{e^r - d}{u-d}}_{=: p_u}\right)$$

$$\Rightarrow C = e^{-r}(p_d C_d + p_u C_u)$$

note: • $p_d = \frac{u - e^r}{u-d} = \frac{u - d + d - e^r}{u-d} = 1 - p_u$

• we assumed $d < e^r < u \Rightarrow 0 < p_d < 1$ and $0 < p_u < 1$

$\Rightarrow$ p_u, p_d are called risk-neutral probabilities

What would be the expectation value of stock price at T under probabilities p_u, p_d?

$$\Rightarrow \mathbb{E}(S(T)_{p_u, p_d}) = p_u Su + p_d Sd$$

$$= \left(\frac{e^r - d}{u-d}\right) Su + \left(\frac{u-e^r}{u-d}\right) Sd$$

$$= S\left(\frac{(e^r-d)u + (u-e^r)d}{u-d}\right)$$

$$= e^r S$$; i.e., expected rate of return = riskless rate r (under risk-neutral probabilities)

- remarkable: here result C is independent of probabilities of stock price model (10% chance going up, 90% down $\Rightarrow$ same price)