2.6 Central Limit Theorem

Session 11
Oct. 10, 2019

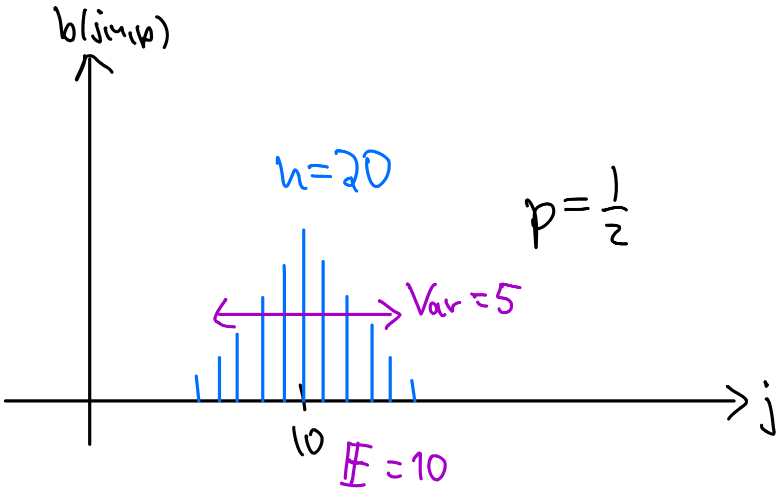
Binomial distribution:

"up" with probability p, "down" with probability $1-p$

$b(j,n,p)$ = probability of j up's in n trials

$$b(j,n,p) = \binom{n}{j} p^j (1-p)^{n-j} \quad , \quad \binom{n}{j} = \frac{n!}{(n-j)!\,j!}$$

note/recall:
- $\sum_{j=0}^{n} b(j,n,p) = \sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} = (p + (1-p))^n = 1$
- $\mathbb{E}(j) = np$
- $\mathrm{Var}(j) = np(1-p)$



center the distribution by shifting $Y_j = j - \mathbb{E}(j) = j - np$

$\Rightarrow \mathbb{E}(Y_j) = \mathbb{E}(j) - np = np - np = 0$

normalize variance by setting $X_j = \dfrac{j - np}{\sqrt{np(1-p)}}$

$\mathrm{Var}(\lambda X) = \lambda^2 \mathrm{Var}(X)$

$\Rightarrow \mathbb{E}(X_j) = 0$ and $\mathrm{Var}(X_j) = \dfrac{1}{np(1-p)} \mathrm{Var}(j - np) = 1$

cumulative distribution = probability for a or fewer up's: $\sum_{j=0}^{a} b(j,n,p) = \sum_{j=0}^{a} b(j,n,p) \underbrace{\Delta j}_{=1}$

$$j = \sqrt{np(1-p)}\, x + np \quad \Big| \quad \Delta j = \sqrt{np(1-p)}\, \Delta x$$

In the limit $n\to\infty$ we would expect

$$\sum_{j=0}^{a} b(j,n,p)\,\Delta j = \sum_{x=np}^{\frac{a-np}{\sqrt{np(1-p)}}} b\left(\sqrt{np(1-p)}\,x+np,n,p\right)\sqrt{np(1-p)}\,\Delta x \longrightarrow \int_{-\infty}^{\tilde a} \varphi(x)\,dx$$

some limit fct.

Central Limit Theorem (CLT) for binomial distribution:

$$\sqrt{np(1-p)}\; b\left(\sqrt{np(1-p)}\,x+np, n, p\right) \xrightarrow{n\to\infty} \varphi(x) \quad \text{pointwise,}$$

where $\varphi(x)$ is the Gaussian with mean 0 and variance 1, i.e.,

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} =: \mathcal{N}(0,1)$$

mean variance

note: • here we have pointwise convergence, whereas usually CLT gives convergence of cumulative distribution fct.

polar coordinates

• $$\left(\int_{-\infty}^{\infty}\varphi(x)dx\right)^2 = \int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\,\frac{1}{2\pi}e^{\frac{-x^2-y^2}{2}} = \int_0^{\infty} r\,dr\int_0^{2\pi}d\varphi\, e^{-\frac{r^2}{2}}\frac{1}{2\pi}$$

$$= \int_0^\infty dr\, r e^{-\frac{r^2}{2}} = -e^{-\frac{r^2}{2}}\Big|_0^\infty = 1 \text{ , indeed normalized}$$

• check: $$\mathbb{E}(x) = \int_{-\infty}^{\infty} x\varphi(x)dx = \underbrace{\int_{-\infty}^{0} x\varphi(x)dx}_{x\to -x} + \int_0^\infty x\varphi(x)dx = 0$$

$$= -\int_0^{-\infty} x\varphi(x)dx = -\int_0^\infty(-x)\varphi(-x)(-dx) = -\int_0^\infty x\varphi(x)dx$$

(x odd fct., $\varphi(x)$ even ($\varphi(-x)=\varphi(x)$))

• check: $\mathrm{Var}(X) = \int_{-\infty}^{\infty} x^2 \varphi(x)\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2}}\,dx$

integration by parts:
$\int fg = fG - \int f'G$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \left(x e^{-\frac{x^2}{2}}\right) dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(x\left(-e^{-\frac{x^2}{2}}\right)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \left(-e^{-\frac{x^2}{2}}\right) dx \right)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx$$

$$= 1$$

Proof of Thm.: direct computation using

• Stirling approximation: $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ (see next HW)

$$\ln n! = \sum_{i=1}^{n} \ln i \approx \int_1^n \ln x\,dx = \int_1^n 1\cdot \ln x\,dx = \ln x \cdot x\Big|_1^n - \int_1^n x\frac{1}{x}\,dx$$

$\ln ab = \ln a + \ln b$

$$= n\ln n - \int_1^n dx = n\ln n - n + 1 \approx n\ln n - n$$

$$\Rightarrow n! \approx e^{n\ln n - n} = n^n e^{-n} = \left(\frac{n}{e}\right)^n$$

• Taylor expansion

2.7 Black-Scholes Formula

recall: option price for European calls:

$$C = e^{-rT} \sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \max(0, Su^j d^{n-j} - K)$$

$$= e^{-rT}\, \mathbb{E}(\text{payoff}) \qquad (r = \text{period interest rate},\ K = \text{strike price})$$

when is payoff $\neq 0$, i.e., $Su^j d^{n-j} - K > 0$?

$$\left(\frac{u}{d}\right)^j > \frac{K}{Sd^n} \quad \Rightarrow \quad j > \frac{\ln \frac{K}{Sd^n}}{\ln \frac{u}{d}} = \frac{\ln \frac{K}{S} - n \ln d}{\ln \frac{u}{d}} =: a$$

$$\Rightarrow C = e^{-rT} \sum_{j=a}^{n} \binom{n}{j} p^j (1-p)^{n-j} \left(Su^j d^{n-j} - K \right)$$

$$= S \sum_{j=a}^{n} \binom{n}{j} \left(pue^{-r\frac{T}{n}}\right)^j \left((1-p)de^{-r\frac{T}{n}}\right)^{n-j} - Ke^{-rT} \sum_{j=a}^{n} \binom{n}{j} p^j (1-p)^{n-j}$$

recall: $p = \dfrac{e^{r\frac{T}{n}} - d}{u - d} \Rightarrow 1 - p = \dfrac{u - d - e^{r\frac{T}{n}} + d}{u - d} = \dfrac{u - e^{r\frac{T}{n}}}{u - d}$

$$\Rightarrow (1-p)de^{-r\frac{T}{n}} = \frac{u - e^{r\frac{T}{n}}}{u - d}\, de^{-r\frac{T}{n}} = \frac{ude^{-r\frac{T}{n}} - d}{u - d} = \frac{ude^{-r\frac{T}{n}} - u + u - d}{u - d}$$

$$= 1 - pue^{-r\frac{T}{n}}$$

$$\Rightarrow C = S \sum_{j=a}^{n} b(j; n, pue^{-r\frac{T}{n}}) - Ke^{-rT} \sum_{j=a}^{n} b(j; n, p)$$

next: use calibration, compute p and a, take $\lim_{n \to \infty}$ and use CLT