4. Black-Scholes Equation and Finite Difference Schemes

Session 19
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4.1 Derivation of the Black-Scholes Equation

Let the stock price process be geom. BM: $dS = \mu S\,dt + \sigma S\,dW$

solution: $S(t) = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)}$

Let $C(S,t)$ be the price of an option

Itô's Lemma gives $dC = \left(\frac{\partial C}{\partial t} + \frac{\partial C}{\partial S}\mu S + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 \right) dt + \frac{\partial C}{\partial S}\sigma S\,dW$

(recall Itô: $dX = f\,dt + g\,dW$ then $dF(X,t) = \left(\frac{\partial F}{\partial t} + f\frac{\partial F}{\partial x} + \frac{1}{2}g^2\frac{\partial^2 F}{\partial x^2} \right) dt + \frac{\partial F}{\partial x} g\,dW$)

Merton's trick: consider a portfolio of C and S that eliminates risk

value of portfolio $\Pi = \alpha C + \beta S$ for some α, β

$\Rightarrow d\Pi = \alpha\,dC + \beta\,dS$

$$= \alpha\left(\frac{\partial C}{\partial t} + \frac{\partial C}{\partial S}\mu S + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 \right) dt + \alpha\frac{\partial C}{\partial S}\sigma S\,dW$$

$$+ \beta\mu S\,dt + \beta\sigma S\,dW$$

Choosing $\beta = -\alpha\frac{\partial C}{\partial S}$ eliminates the dW terms, i.e., for $\Pi = \alpha\left(C - \frac{\partial C}{\partial S}S\right)$ we get

$$d\Pi = \alpha\left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2} \right) dt$$

$\Rightarrow$ no uncertainty anymore, so the portfolio has to grow with riskless rate r:

$$d\Pi = \Pi r dt \quad (\text{s.t. } \Pi(t) = \Pi(0) e^{rt})$$

$$= \alpha \left(rC - rS \frac{\partial C}{\partial S} \right) dt$$

Comparing the expressions gives:

$$\boxed{\frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS \frac{\partial C}{\partial S} = rC \quad \text{Black-Scholes Equation}}$$

Notes:

and other parameters through initial conditions

- independent of μ! (option price only depends on volatility!)
- by a change of variables this eq. can be transformed into a heat equation

$$\frac{\partial u}{\partial t} = c \frac{\partial^2 u}{\partial x^2} \quad (\text{or } c \Delta u \text{ for higher dim.})$$

- backward drift-diffusion equation:

European calls, strike price

↳ we specify $C(S,T) = \text{payoff} \stackrel{\downarrow}{=} \max(0, S-K)$

↳ we solve for $C(S,0)$

4.2 Connection between Black-Scholes eq. and Formula

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC$$

European call: $C(S,T) = \max(S-K,0)$

with boundary condition $C(0,t)=0$

we do several changes of variables to reduce it to the heat eq.

$$C(S,t) = B(S,\tau)e^{-r\tau}K \qquad , \tau = T-t$$

$\Rightarrow$ in terms of B: $-\frac{\partial B}{\partial \tau} + rS\frac{\partial B}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 B}{\partial S^2} = 0$

$$B(0,\tau) = 0 \quad , B(S,0) = \max\left(\frac{S}{K}-1,0\right)$$

to remove first derivative: $D(x,\tau) = B(S,\tau)$, $x = \frac{S}{K}e^{r\tau}$

$$\Rightarrow \quad -\frac{\partial D}{\partial \tau} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 D}{\partial x^2} = 0$$

to remove σ: $H(x,u) = D(x,\tau)$, $u = \sigma^2\tau$

$$\Rightarrow -\frac{\partial H}{\partial u} + \frac{1}{2}x^2\frac{\partial^2 H}{\partial x^2} = 0$$

to remove x^2-prefactor: $\Theta(z,u) = H(x,u)$, $z = \frac{u}{2} + \ln x$

$\Rightarrow \boxed{-\frac{\partial \Theta}{\partial u} + \frac{1}{2}\frac{\partial^2 \Theta}{\partial z^2} = 0}$ heat equation , $\Theta(z,0) = \max(1-e^{-z},0)$

How to solve the heat eq.?

inverse Fourier transform: $\theta(z,u) = \frac{1}{\sqrt{2\pi}} \int e^{-ikz} \hat{\theta}(k,u)\,dk$

plug into eq:

$$\frac{1}{\sqrt{2\pi}} \int e^{-ikz} \frac{\partial \hat{\theta}(k,u)}{\partial u}\,dk = \frac{1}{\sqrt{2\pi}} \int e^{-ikz} \left(\frac{1}{2}(-ik)^2\right) \hat{\theta}(k,u)\,dk$$

solve $\frac{\partial \hat{\theta}(k,u)}{\partial u} = -\frac{k^2}{2} \hat{\theta}(k,u)$

$$\Rightarrow \quad \hat{\theta}(k,u) = e^{-\frac{k^2}{2}u} \hat{\theta}(k,0)$$

$$\Rightarrow \theta(z,u) = \frac{1}{\sqrt{2\pi}} \int e^{-ikz} e^{-\frac{k^2}{2}u} \underbrace{\hat{\theta}(k,0)}_{= \frac{1}{\sqrt{2\pi}} \int e^{iky} \theta(y,0)\,dy} dk$$

$$= \frac{1}{2\pi} \int \underbrace{\left[\int e^{-ik(z-y)} e^{-\frac{k^2}{2}u}\,dk \right]}\theta(y,0)\,dy$$

$$= \int dk\, e^{-\frac{u}{2}\left[k^2 + 2\frac{ik(z-y)}{u} + \left(\frac{i(z-y)}{u}\right)^2 - \left(\frac{i(z-y)}{u}\right)^2\right]}$$

$$= \int dk\, e^{-\frac{u}{2}\left(k + i\frac{z-y}{u}\right)^2} e^{-\frac{(z-y)^2}{2u}}$$

$$= e^{-\frac{(z-y)^2}{2u}} \underbrace{\int dk\, e^{-\frac{u}{2}k^2}} \qquad k = \ell\sqrt{\frac{2}{u}}$$

$$= \sqrt{\frac{2}{u}} \int d\ell\, e^{-\ell^2}$$

$$= \sqrt{\frac{2\pi}{u}}$$

$$\Rightarrow \theta(z,u) = \frac{1}{\sqrt{2\pi u}} \int e^{-\frac{(z-y)^2}{2u}} \theta(y,0)\, dy$$

$\Rightarrow$ with B-S initial cond. $\theta(y,0) = \max(1-e^{-y}, 0)$ we get

$$\theta(z,u) = \frac{1}{\sqrt{2\pi u}} \int_0^\infty e^{-\frac{(z-y)^2}{2u}} (1-e^{-y})\, dy$$

substituting back our changes of variables we get B-S formula.