Stochastic Methods Lab, Jacobs University Bremen, Fall 2020

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Session 5
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1.3 Bonds

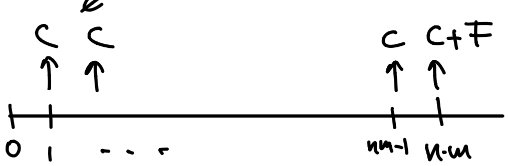
bond issuer (borrower) pays interest and final payment to bond holder (lender, buyer)

↳ usually for long-term debts, e.g. issued by governments (but also companies)

↳ bonds are fully repaid at maturity date

cashflow for level-coupon bond:

payments at end of payment period



present value / price $P = \sum_{i=1}^{n\cdot m} \frac{C}{(1+\frac{r}{m})^i} + \frac{F}{(1+\frac{r}{m})^{nm}}$

where:
- C = coupon payments
- r = interest rate
- F = par value
- n = # of periods (usually years)
- m = # payments per period
- with $C = \frac{F \cdot c}{m}$, c = coupon rate

$$\Rightarrow P = F \left(\sum_{i=1}^{nm} \frac{c/m}{(1+\frac{r}{m})^i} + \frac{1}{(1+\frac{r}{m})^{nm}} \right)$$

- P, C (or c), F, n, m determine the "bond contract"
  given these values, the r = IRR = yield to maturity

Ex.: $\underbrace{20}_{n}$ year, $\underbrace{9\%\text{ bond}}_{\text{coupon rate}}$, $\underbrace{\text{semiannual compounding}}_{m=2}$, interest rate $r = 8\%$

$$\text{price } P = F\left(\sum_{i=1}^{40} \frac{0.09/2}{1.04^i} + \frac{1}{(1.04)^{40}}\right) = 1.099\cdot F$$

=> this bond sells at 109.9% of par

e.g., par value $F = 1000\,\$$ => $P = 1099\,\$$ and $C = 45\,\$$.

Using the geometric series, we find (let's do $m=1$ here):

$$P = F\left(c\underbrace{\sum_{i=1}^{n} \frac{1}{(1+r)^i}} + \frac{1}{(1+r)^n}\right)$$

$$= -1 + \frac{1-(1+r)^{-n-1}}{1-(1+r)^{-1}} = \frac{-1+(1+r)^{-1}+1-(1+r)^{-n-1}}{1-(1+r)^{-1}} = \frac{(1+r)^{-1}-(1+r)^{-n-1}}{1-(1+r)^{-1}} \overset{\frac{1+r}{1+r}}{=} \frac{1-(1+r)^{-n}}{r}$$

$$= F\left(\frac{c}{r}\left(1-(1+r)^{-n}\right) + (1+r)^{-n}\right)$$

$$= F\left(\frac{c}{r} + \frac{1-\frac{c}{r}}{(1+r)^n}\right)$$

Note: often one uses $c=0$ ($C=0$), these are called **zero-coupon bonds**.

terminology:

- $c = r$, then $P = F$, and "the bond sells at par"
- $c > r$, then $P > F$, and "the bond sells above par"
- $c < r$, then $P < F$, and "the bond sells at a discount"

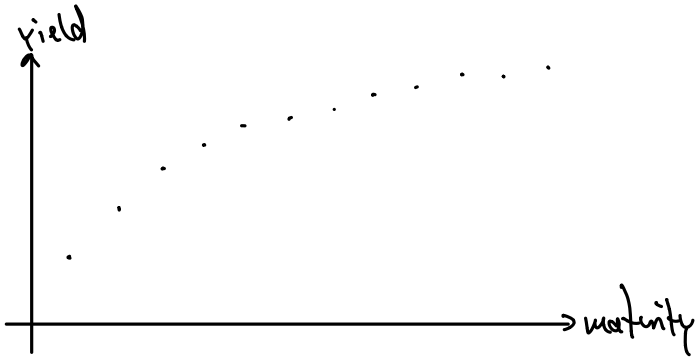
1.4 Spot Rates

yield / interest rates should be different for different maturities

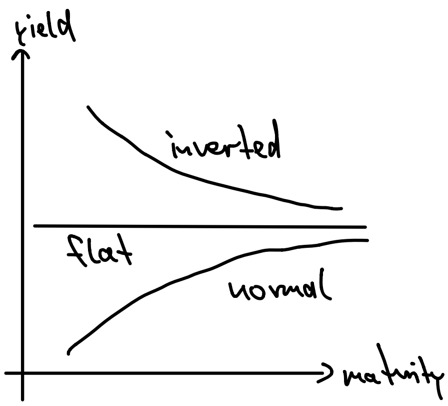
usually: longer comitment (maturity) $\Rightarrow$ higher interest

this phenomenon is called "term structure"

yield curve



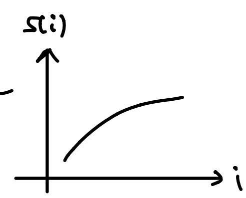
types of curves:



Spot rate $S(i)$ = yield to maturity of i-period zero-coupon bond

(they follow the relation $P = \frac{F}{(1+S(i))^i}$)

$\Rightarrow$ spot rate curve is the zero-coupon bond yield curve



Suppose the $S(i)$ are given by some standard, say, in the US the US-treasury zero-coupon bonds, then a better zero-coupon bond price formula would be

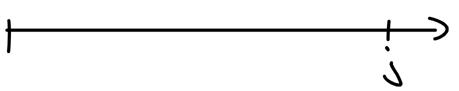
$$P = \sum_{i=1}^{n} \frac{C}{(1+S(i))^i} + \frac{F}{(1+S(n))^n} \qquad (m=1 \text{ here})$$

note: $d(i) := \frac{1}{(1+S(i))^i}$ are called discount factors

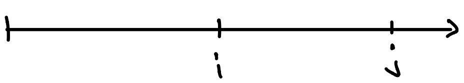
note: risky bonds should be cheaper, this is often taken into account by adding a "static spread" s : $(1+S(i))^{-i} \longrightarrow (1+s+S(i))^{-i}$

side remark: there is also the concept of $\boxed{\text{forward rates}}$

consider 0-coupon bond



$$FV_j = P(1+S(j))^j$$



$$FV_i = P(1+S(i))^i$$

$$\Rightarrow FV_j = FV_i\,(1+S(i,j))^{j-i}$$
$$= P(1+S(i))^i (1+S(i,j))^{j-i}$$

$S(i,j) = (j-i)$-period spot rate i periods from now (unknown)

(there are many models for $S(i,j)$)

Sometimes the simple model of (implied) forward rates is used:

$f(i;j)$ = model for $S(i;j)$ based on

$$(1+S(j))^{j} = (1+S(i))^{i}\,(1+f(i;j))^{j-i}$$

$$\Rightarrow f(i;j) = \left(\frac{(1+S(j))^{j}}{(1+S(i))^{i}}\right)^{\frac{1}{j-i}} - 1$$