Jacobs University
Spring 2020

May 9, 2020

Calculus and Linear Algebra II

Homework 6

Since the TAs also have exams, this sheet will not be graded. A solution will be provided on moodle.

Problem 1 [**7 points**]
A general unitary 2×2 matrix can be written in the form

$$U = \begin{pmatrix} a & b \\ -e^{i\varphi}\overline{b} & e^{i\varphi}\overline{a} \end{pmatrix},$$

where $a, b \in \mathbb{C}$ such that $|a|^2 + |b|^2 = 1$, and $\varphi \in [0, 2\pi)$. For matrices of this form, check explicitly the following properties of unitary matrices that we stated in class: that the columns are orthonormal, that the rows are orthonormal, and that $U^\dagger = U^{-1}$. Also compute the determinant of U, and verify that it has absolute value one. (The computation of eigenvalues and eigenvectors is a bit lengthy, so let us skip it here.)

Problem 2 [**7 points**]
Show that the matrix

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}$$

is unitary, and compute all the eigenvalues and eigenvectors.

Problem 3 [**4 points**]
In class, we briefly discussed the Jordan normal form. Just to get an idea how it can be useful, let us consider the matrix

$$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},$$

with some $\lambda \in \mathbb{R}$. Find a general formula for J^k for any $k \in \mathbb{N}$. *Hint: Start by computing J^2, J^3, and then find the general pattern. It is also a good exercise to write down a nice clear proof of the formula using induction.*

Problem 4 [**8 points**]
We consider the function $f(x, y) = 4x^2 + 4y^2 + x^4 - 6x^2y^2 + y^4$. Compute the gradient of f, all critical points, and the Hessian matrix. For each critical point, evaluate the Hessian matrix and compute the eigenvalues. Conclude from this whether the critical points are minima, maxima, or saddle points.

Problem 5 [4 points]
Show that for anti-Hermitian matrices all eigenvalues are purely imaginary or zero, i.e., that they can be written as $\lambda = i\alpha$ with $\alpha \in \mathbb{R}$.

Bonus Problem [8 points]
Let us consider the Hermitian matrix

$$H = \frac{\pi}{4} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}.$$

Compute e^{iH} in two different ways: first, by diagonalizing H, and second, by directly computing the power series (think about what H^2 is, and deduce from that what H^k is). Verify that the resulting matrix is unitary.