Jacobs University
Spring 2020

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Calculus and Linear Algebra II

Quiz 3

Instructions:

- Do all the work on this quiz paper.
- Show your work, i.e., write down the steps of your solution cleanly and readable.
- Electronic devices and notes are not allowed.

Name: Solutions

Problem 1 [7 points]

Let $f(x, y) = x\cos(y) - y^2$.

(a) Compute the directional derivative of f at $(0, 0)$ in direction $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

Solution:

We find

$$(\nabla f)(x, y) = \begin{pmatrix} \cos(y) \\ -x\sin(y) - 2y \end{pmatrix} \quad \Rightarrow \quad (\nabla f)(0, 0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Then the directional derivative at $(0, 0)$ in direction $\vec{a} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ is

$$(D_{\vec{a}}f)(0, 0) = (\nabla f)(0, 0) \cdot \vec{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}.$$

(b) Now assume that $x(t) = t^2$ and $y(t) = t^2$. Compute $\frac{d}{dt}f(x(t), y(t))$.

Solution:

We can use the chain rule:

$$\begin{aligned} \frac{d}{dt}f(x(t), y(t)) &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ &= \cos(y)2t + (-x\sin(y) - 2y)2t \\ &= 2t\cos(t^2) - 2t^3\sin(t^2) - 4t^3. \end{aligned}$$

(c) Next, compute $\frac{d}{dx}\int_0^{\pi/2} f(x, y)dy$.

Solution:

We can use the Leibniz integral rule:

$$\frac{d}{dx}\int_0^{\pi/2} f(x, y)dy = \int_0^{\pi/2} \frac{\partial f(x, y)}{\partial x}dy = \int_0^{\pi/2} \cos(y)dy = \sin(y)\Big|_0^{\pi/2} = 1.$$

Problem 1 (extra space)

Problem 2 [8 points]

(a) Is the differential $df = e^{-x+2y^2}(-dx + 4ydy)$ exact or inexact?

Solution:

If $df = Adx + Bdy$, we need to check the condition $\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$. Here, we find

$$\frac{\partial}{\partial y}\Big(-e^{-x+2y^2}\Big) = -4ye^{-x+2y^2}$$
$$\frac{\partial}{\partial x}\Big(4ye^{-x+2y^2}\Big) = -4ye^{-x+2y^2},$$

so the differential is indeed exact. In fact, $f = e^{-x+2y^2}$ has the differential df above.

(b) List what different types of critical points there are for a function $f : \mathbb{R}^n \to \mathbb{R}$.

Solution:

The critical points can be

- maxima (local or global)
- minima (local or global)
- saddle points
- degenerate points
- bonus point if you mention monkey saddle

(c) Find all critical points of $f(x, y) = e^{x^2-y^2}$. What kind of critical points are they? (Try to answer by thinking about what this function looks like.)

Solution:

We have to check the condition $\nabla f = 0$. We find

$$(\nabla f)(x, y) = \begin{pmatrix} 2x \\ -2y \end{pmatrix} e^{x^2-y^2}.$$

This is zero only if $(x, y) = (0, 0)$, i.e., this is the only critical point. Clearly, the function has a minimum if we go along the x-axis, and a maximum if we go along the y-axis, so the critical point is a saddle point.

(d) [Only if you are already done with the other problems and are bored.] Find all critical points of $f(x, y) = xy(12 - 3x - 4y)$.

Solution:

Checking the condition $\nabla f = 0$ yield the four critical points $(0, 0), (4, 0), (0, 3), (\frac{4}{3}, 1)$.

Problem 2 (extra space)