Jacobs University
Spring 2020

April 29, 2020

Calculus and Linear Algebra II

Quiz 6

Instructions:

- Do all the work on this quiz paper.
- Show your work, i.e., write down the steps of your solution cleanly and readable.
- Electronic devices and notes are not allowed.

Name: Solutions

Problem 1 [8 points]

Compute the determinants of the following matrices. Look at these matrices carefully and try to infer what the determinant is with as little computation as possible. Also state for each matrix whether it is invertible or not.

(a)

$$\begin{pmatrix} 0 & 0 & 1 \\ 3 & 1 & 2 \\ 1 & 3 & 2 \end{pmatrix},$$

(b)

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 7 & 9 & 3 & 5 \\ 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix},$$

(c)

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 566 & 500 \\ 0 & 0 & 1 & 343 \\ 0 & 0 & 0 & 4000 \end{pmatrix},$$

(d)

$$\begin{pmatrix} 250 & 876865 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 435352 & 2 \\ 1 & 0 \end{pmatrix}$$

(Here it is asked for the determinant of the product of these two matrices, and whether or not this product has an inverse).

Solution:

(a) A Laplace expansion by the first row gives us that

$$\det \begin{pmatrix} 0 & 0 & 1 \\ 3 & 1 & 2 \\ 1 & 3 & 2 \end{pmatrix} = \det \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} = 3 \times 3 - 1 \times 1 = 8.$$

The matrix is invertible, since the determinant is non-zero.

(b) The determinant is zero since the first and the last row are the same. The matrix is not invertible (it is singular), since the determinant is zero.

(c) For an upper triangular matrix, the determinant is the product of the diagonal entries. Here, the determinant is equal to $1 \times 2 \times 1 \times 4000 = 8000$. The matrix is invertible, since the determinant is non-zero.

(d) The determinant of the product of two matrices is the product of the determinants. Thus,

$$\begin{aligned}\det\left[\begin{pmatrix} 250 & 876865 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 435352 & 2 \\ 1 & 0 \end{pmatrix}\right] &= \det\left[\begin{pmatrix} 250 & 876865 \\ 0 & 1 \end{pmatrix}\right]\det\left[\begin{pmatrix} 435352 & 2 \\ 1 & 0 \end{pmatrix}\right] \\ &= 250 \times (-2) \\ &= -500.\end{aligned}$$

The matrix is invertible, since the determinant is non-zero.

Problem 2 [7 points]

Compute all eigenvalues and the associated eigenvectors/eigenspace of the matrix

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

and determine what the algebraic and geometric multiplicity of each eigenvalue is.

Solution:

The characteristic polynomial is

$$\det \begin{pmatrix} 2-\lambda & 1 & -1 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{pmatrix} = (2-\lambda)^2(1-\lambda).$$

Thus, there are two eigenvalues: $\lambda_1 = 1$ has algebraic multiplicity 1, and $\lambda_2 = 2$ has algebraic multiplicity 2. To compute the eigenvalues we have to solve

(a)

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2-\lambda_1 & 1 & -1 \\ 0 & 2-\lambda_1 & 1 \\ 0 & 0 & 1-\lambda_1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

The last row tells us nothing, the second row tell us that $y = -z$, and the first row that $x = 2z$. Thus, all eigenvectors to λ_1 are

$$\begin{pmatrix} 2z \\ -z \\ z \end{pmatrix}, \text{ for any } z \neq 0.$$

We could thus write the eigenspace as

$$E_{\lambda_1} = \left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

This is one-dimensional, so the geometric multiplicity is 1.

(b)

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2-\lambda_2 & 1 & -1 \\ 0 & 2-\lambda_2 & 1 \\ 0 & 0 & 1-\lambda_2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

The second and third rows tell us that $z = 0$. The first row tells us that $y = z$. Thus, x is free, and all eigenvectors to λ_2 are

$$\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}, \text{ for any } x \neq 0.$$

We could thus write the eigenspace as

$$E_{\lambda_1} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

This is one-dimensional, so the geometric multiplicity is 1.

Note that for λ_2 the algebraic and geometric multiplicities are not the same, thus the matrix is not diagonalizable.