Calculus and Linear Algebra II

Session 1
Feb. 3, 2020

Prof. Sören Petrat

Research area: Mathematical Physics

Organization:

- basic information: campusnet
- up-to-date info: class website
- class: Mon/Wed, 14:15 – 15:30 (Conrad-Naber Lecture Hall)
- if you have questions: ask me after class (won't be able to answer all emails)
- if you need prerequisite waiver: after class

- grade only based on final exam
- books: see website
- lecture notes: see website
- TAs: Dzmitry, Abhik, Cyrine, Miruna

- tutorials:
  - questions, individual help with exercises, practice
  - 3 time slots, tbd, starting next week
  - really important!

- exercises:
  - homework sheets (see website, moodle)
    - ↳ hand in after class on due date
    - ↳ or put in mailbox at entrance of Research I
  - moodle exercises
  - biweekly quizzes
- from each pool of exercises, 2 will be on final exam!

1. Some Extra Part I Topics

1.1 Binomial Expansion

we would like to expand $(a+b)^n$

<u>Ex.:</u>
$$(a+b)^0 = 1$$
$$(a+b)^1 = a+b$$
$$(a+b)^2 = a^2+2ab+b^2$$
$$(a+b)^3 = a^3 + \underbrace{3}\, a^2 b + 3ab^2 + b^3$$

= # of possibilities to distribute 2a's and 1 b into 3 slots : aab
aba
baa

Generally:
$$(a+b)^{n+1} = (a+b)^n\,(a+b)$$
$$= \left(a^n + \dots + C(n,k)\, a^{n-k} b^k + \dots + b^n\right)(a+b)$$
$$= a^{n+1} + \dots + \underbrace{\Big(C(n,k-1) + C(n,k)\Big)}_{= C(n+1,k)} a^{n+1-k} b^k + \dots + b^{n+1}$$

General pattern: Pascal triangle

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & \underbrace{1 & + & 3} & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
 & & \vdots & & & & \vdots & &
\end{array}$$

$n \downarrow$

$\longrightarrow$
$k = 0, 1, \dots$

Formula: $C(n,k) = \boxed{\binom{n}{k} := \frac{n!}{(n-k)!\,k!}} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{1\cdot 2\cdot 3\cdots k}$

"n factorial"

defined as

"n choose k" (sometimes: $C(n,k) = {}^{n}C_{k}$)

Theorem: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \dots + \binom{n}{n} b^n$

sum from $k=0$ to n

Binomial expansion

Note: • $0! := 1$, so $\binom{n}{0} = \frac{n!}{n!\,0!} = 1$, $\binom{n}{n} = \frac{n!}{(n-n)!\,n!} = 1$

• symmetry: $\binom{n}{k} = \frac{n!}{(n-k)!\,k!} = \binom{n}{n-k}$

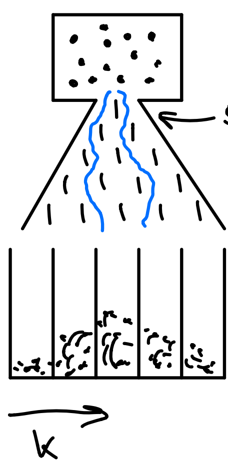
• property from above:

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(n-k+1)!\,(k-1)!}\cdot\frac{k}{k} + \frac{n!}{(n-k)!\,k!}\cdot\frac{(n-k+1)}{(n-k+1)}$$

$$= \frac{n!\,k}{(n-k+1)!\,k!} + \frac{n!\,(n-k+1)}{(n-k+1)!\,k!}$$

$$= \frac{n!\,(k+n-k+1)}{(n-k+1)!\,k!} = \frac{(n+1)!}{(n+1-k)!\,k!} = \binom{n+1}{k} \checkmark$$

Different perspective: Galton board



50-50 chance to go left or right (as in coin toss)

Gauss distribution! (in a continuous limit) how many paths end up in box k?

$\hookrightarrow$ need $n-k$ times left, k times right $\Rightarrow \binom{n}{k}$ paths

$\Rightarrow$ probability for landing in box k: $\binom{n}{k} \quad p^k (1-p)^{n-k} =: \mathbb{P}(n,k)$

$\hookrightarrow p \in (0,1)$ = probability to go right

(before we had $p=\frac{1}{2}$)

$\Rightarrow$ Probability to end up somewhere: $\sum_{k=0}^{n} \mathbb{P}(n,k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (p+1-p)^n = 1$ ✓

Question: What is expectation value $\mathbb{E} = \sum_{k=0}^{n} k\, \mathbb{P}(n,k)$?

$\hookrightarrow \mathbb{E} = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} = (1-p)^n \sum_{k=0}^{n} k \binom{n}{k} \Big(\underbrace{\frac{p}{1-p}}_{:=x}\Big)^k$

$\hookrightarrow$ need to compute $\sum_{k=0}^{n} k \binom{n}{k} x^k$

$\hookrightarrow$ trick: $x \cdot \frac{d}{dx} \underbrace{\sum_{k=0}^{n} \binom{n}{k} x^k} = \sum_{k=0}^{n} k \binom{n}{k} x^k$

$= \sum_{k=0}^{n} \binom{n}{k} x^k 1^{n-k}$

$= (x+1)^n$

$\Rightarrow x \frac{d}{dx} (x+1)^n = x\, n\, (x+1)^{n-1}$

$\Rightarrow \mathbb{E} = (1-p)^n\, x\, n\, (x+1)^{n-1} = \underbrace{(1-p)^n \left(\frac{p}{1-p}\right)}_{(1-p)^{n-1} p} n \Big(\underbrace{\frac{p}{1-p}+1}\Big)^{n-1} = n \cdot p$

$= \left(\frac{p}{1-p} + \frac{1-p}{1-p}\right)^{n-1} = \left(\frac{1}{1-p}\right)^{n-1}$

similarly: variance $\sum_{k=0}^{n} k^2\, \mathbb{P}(n,k) - \mathbb{E}^2$ (see HW)