More on improper integrals ...

Definition:

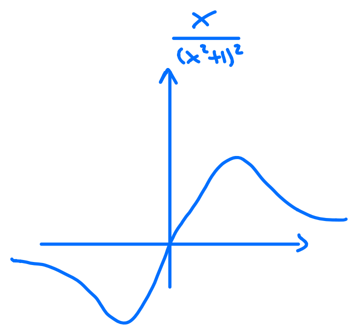
$\int_{-\infty}^{\infty} f(x)\,dx := \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$, if both integrals exist, is also called improper integral.

Ex.: • $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(x^2+1)^2}\,dx = \int_{-\infty}^{0} \frac{x}{(x^2+1)^2}\,dx + \int_{0}^{\infty} \frac{x}{(x^2+1)^2}\,dx$

$$= -\frac{1}{2}\,\frac{1}{x^2+1}\Big|_{-\infty}^{0} - \frac{1}{2}\,\frac{1}{x^2+1}\Big|_{0}^{\infty}$$

$$= -\frac{1}{2} - 0 + \left(0 - \left(-\frac{1}{2}\right)\right)$$

$$= 0$$



• $\displaystyle\int_{-\infty}^{\infty} x\,dx = \int_{-\infty}^{0} x\,dx + \int_{0}^{\infty} x\,dx$ does not exist!

(note: $\displaystyle\int_{-\infty}^{\infty} x\,dx \neq \lim_{b\to\infty} \int_{-b}^{b} x\,dx = \lim_{b\to\infty} \frac{x^2}{2}\Big|_{-b}^{b} = 0$)

Definition:

If $f : (a,b] \to \mathbb{R}$ is singular at $a \in \mathbb{R}$ (i.e., $\lim_{\varepsilon\to\infty} f(a+\varepsilon) = \pm\infty$), then

$\displaystyle\int_{a}^{b} f(x)\,dx := \underbrace{\lim_{\varepsilon\to 0^+}}_{\substack{=\lim\limits_{\varepsilon\to 0} \\ \text{with } \varepsilon>0}} \int_{a+\varepsilon}^{b} f(x)\,dx$, if it exists, is also called improper integral.

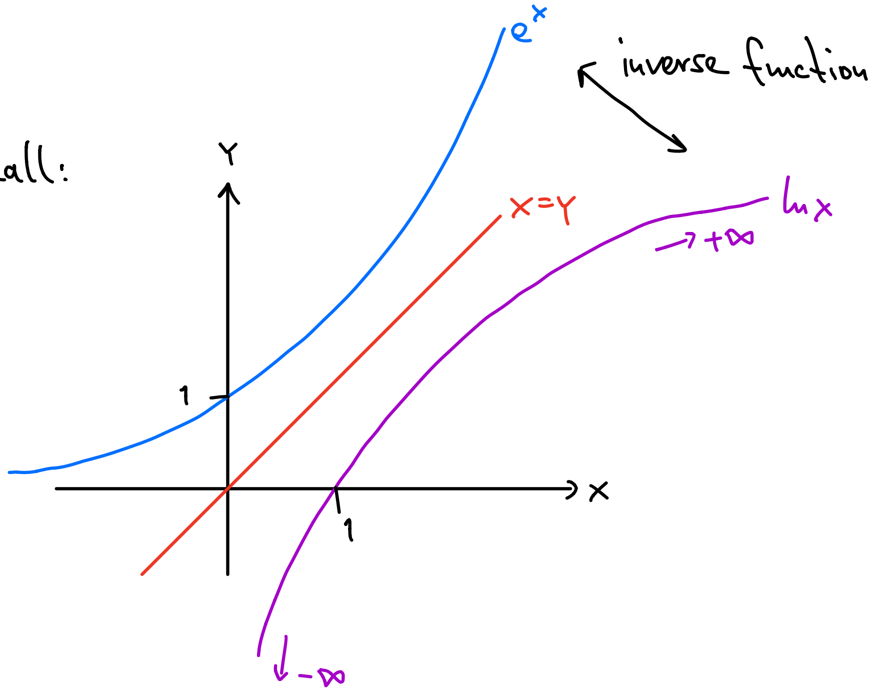
Ex.: for $\alpha > 0$, $\alpha \neq 1$, consider $f(x) = x^{-\alpha}$ $\Rightarrow$ singular at $x=0$

$$\int_0^1 x^{-\alpha}\,dx = \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 x^{-\alpha}\,dx = \lim_{\varepsilon \to 0^+} \left.\frac{x^{-\alpha+1}}{-\alpha+1}\right|_\varepsilon^1$$

$$= \lim_{\varepsilon \to 0^+} \left[\frac{1}{1-\alpha} - \frac{\varepsilon^{1-\alpha}}{1-\alpha}\right] = \begin{cases} \text{divergent if } \alpha > 1 \\ \frac{1}{1-\alpha} \text{ if } 0 < \alpha < 1 \end{cases}$$

note: $\alpha = 1$: $\displaystyle\int_0^1 x^{-1}\,dx = \lim_{\varepsilon \to 0^+} \ln x \Big|_\varepsilon^1 = \lim_{\varepsilon \to 0^+} \big(\underbrace{\ln 1}_{=0} - \ln \varepsilon\big)$ diverges!

recall:



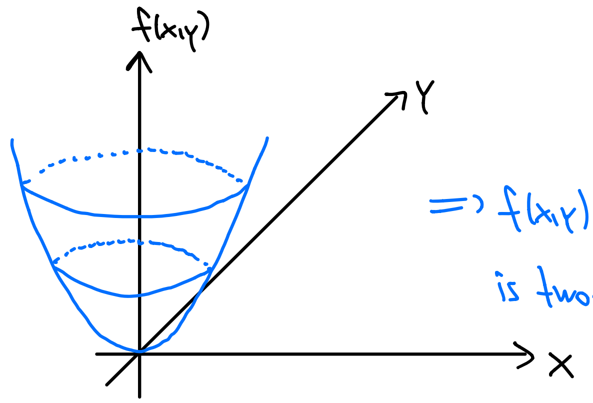
2. Multivariable Calculus: Derivatives

2.1 Partial Derivatives

recall: one-variable function: $f: \mathbb{R} \to \mathbb{R}$, $x \longmapsto f(x)$

now: $f: \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \longmapsto f(x,y)$

Ex.: $f(x,y) = x^2 + y^2$



$\Rightarrow$ $f(x,y)$ is a surface, i.e., its graph is two-dimensional

note: $f(x,y,z)$ would be harder to visualize

In general, we consider $f: \mathbb{R}^n \to \mathbb{R}$, $(x_1, \ldots, x_n) \longmapsto f(x_1, \ldots, x_n)$, called a

real-valued function on $\mathbb{R}^n$.

How do we take derivatives, e.g., of $f(x,y) = x^2 + y^2$

↳ keep y fixed, take derivative in x: $\dfrac{\partial f(x,y)}{\partial x} = 2x$

$\Rightarrow$ usual derivative along x-axis

↳ keep x fixed, take derivative in y: $\dfrac{\partial f(x,y)}{\partial y} = 2y$

$\Rightarrow$ usual derivative along y-axis

Definition: For $f: \mathbb{R}^n \to \mathbb{R}$, we call

$$\frac{\partial f(x_1,\dots,x_n)}{\partial x_i} := \lim_{\Delta x_i \to 0} \frac{f(x_1,\dots,x_i+\Delta x_i,\dots,x_n) - f(x_1,\dots,x_i,\dots,x_n)}{\Delta x_i},$$

if it exists, the i-th **partial derivative** of f.

Ex.: $f(x_1,x_2,x_3) = \sin(x_1)\cdot x_2^2 + e^{x_3}$

$$\Rightarrow \frac{\partial f}{\partial x_1} = \cos(x_1)\, x_2^2$$

$$\frac{\partial f}{\partial x_2} = \sin(x_1)\, 2x_2$$

$$\frac{\partial f}{\partial x_3} = e^{x_3}$$

Note: we can also take higher-order partial derivatives:

↳ e.g., for $f(x,y)$, we can do $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) := \frac{\partial^2 f}{\partial x^2}$,

but also $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) := \frac{\partial^2 f}{\partial x \partial y}$ ("mixed partial derivative"),

or $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) := \frac{\partial^2 f}{\partial y \partial x}$

Ex.: $f(x,y) = \sin(x)\cdot y^3 + x^2$

$$\Rightarrow \frac{\partial f}{\partial x} = \cos(x)\, y^3 + 2x, \quad \frac{\partial f}{\partial y} = \sin(x)\, 3y^2$$

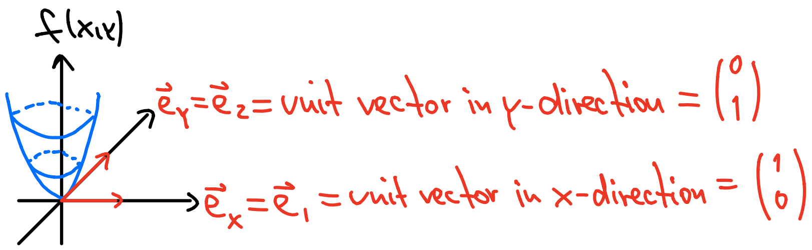
$$\frac{\partial^2 f}{\partial x^2} = -\sin(x)\, y^3 + 2, \quad \frac{\partial^2 f}{\partial y^2} = \sin(x)\, 6y, \quad \underbrace{\frac{\partial^2 f}{\partial x \partial y} = \cos(x)\, 3y^2, \quad \frac{\partial^2 f}{\partial y \partial x} = \cos(x)\, 3y^2}_{\text{equal!}}$$

One can show (no proof here):

Theorem (Clairaut, Schwarz):

If $\frac{\partial^2 f}{\partial x_i \partial x_j}$ and $\frac{\partial^2 f}{\partial x_j \partial x_i}$ are continuous at some point, then $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$ at that point.

Next: recall



let us write (x,y) as a vector: $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, then $f(x,y) = f(\vec{x})$

def. of partial derivative: $\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$

$$= \lim_{\varepsilon \to 0} \frac{f(\vec{x} + \varepsilon \vec{e}_x) - f(\vec{x})}{\varepsilon}$$

In this way, we def. the derivative in any direction $\vec{h} = h_1 \vec{e}_1 + \ldots + h_n \vec{e}_n = \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}$

Definition (directional derivative):

For any $\vec{h} \in \mathbb{R}^n$, we call

$D_{\vec{h}} f := \lim_{\varepsilon \to 0} \frac{f(\vec{x} + \varepsilon \vec{h}) - f(\vec{x})}{\varepsilon}$, if it exists, the derivative in direction $\vec{h}$.

We will see soon how to compute this easily.

Example starting from definition:

How does $f(x,y) = x^2 y + 5x$ change in direction $\vec{h} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$?

$$\Rightarrow D_h f = \lim_{\varepsilon \to 0} \frac{f(\vec{x} + \varepsilon \vec{h}) - f(\vec{x})}{\varepsilon} \qquad \left(\vec{x} + \varepsilon \vec{h} = \vec{x} + \begin{pmatrix} \varepsilon \\ \varepsilon \end{pmatrix} = \begin{pmatrix} x+\varepsilon \\ y+\varepsilon \end{pmatrix} \right)$$

$$= \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon, y+\varepsilon) - f(x,y)}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \underbrace{\frac{(x+\varepsilon)^2 (y+\varepsilon) + 5(x+\varepsilon) - [x^2 y + 5x]}{\varepsilon}}$$

$$= \frac{(x^2 + 2\varepsilon x + \varepsilon^2)(y+\varepsilon) + 5\varepsilon - x^2 y}{\varepsilon}$$

$$= x^2 + 2xy + \underbrace{2x\varepsilon + \varepsilon y + \varepsilon^2}_{\xrightarrow{\varepsilon \to 0} 0} + 5$$

$$\Rightarrow D_h f = x^2 + 2xy + 5$$