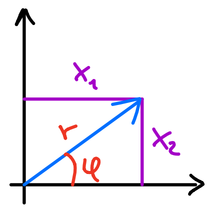
Session 9
Mar 2, 2020

C) Change of variables

e.g., $f\left(x_1(r,\varphi), x_2(r,\varphi)\right)$, $\overbrace{x_1 = r\cos\varphi,\ x_2 = r\sin\varphi}^{\text{polar coordinates}}$ $\quad (r \geq 0, \varphi \in [0, 2\pi))$

$$\Rightarrow \text{e.g. } \frac{\partial f}{\partial r} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial r} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial r}$$



note: usually we are interested in the other way around

$\hookrightarrow r = \sqrt{x_1^2 + x_2^2}, \quad \varphi = \arctan\frac{x_2}{x_1}$

$\hookrightarrow \frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x_1} + \frac{\partial f}{\partial \varphi}\frac{\partial \varphi}{\partial x_1}$, or $\quad \frac{\partial}{\partial x_1} = \frac{\partial r}{\partial x_1}\frac{\partial}{\partial r} + \frac{\partial \varphi}{\partial x_1}\frac{\partial}{\partial \varphi}$

i.e., how we can express derivatives $\partial_{x_1}, \partial_{x_2}$ in terms of $\partial_r, \partial_\varphi$

e.g., here $\frac{\partial r}{\partial x_1} = \frac{x_1}{\sqrt{x_1^2 + x_2^2}} = \frac{r\cos\varphi}{r} = \cos\varphi$

$$\frac{\partial \varphi}{\partial x_1} = \partial_{x_1}\left(\arctan\frac{x_2}{x_1}\right) = \ldots = -\frac{\sin\varphi}{r}$$

$$\Rightarrow \frac{\partial}{\partial x_1} = \cos\varphi\frac{\partial}{\partial r} - \frac{\sin\varphi}{r}\frac{\partial}{\partial \varphi}$$

D) Leibniz integral rule

consider $I(x) = \int_a^b f(x,t)\,dt \overset{\text{keep } x \text{ fixed}}{=} F(x,b) - F(x,a)$ $\quad$ (where $\frac{\partial F}{\partial t} = f$)

then
$$\frac{dI(x)}{dx} = \frac{\partial F(x,b)}{\partial x} - \frac{\partial F(x,a)}{\partial x}$$

$$= \int_a^b \frac{\partial}{\partial t}\left(\frac{\partial F(x,t)}{\partial x}\right)dt$$

Clairaut/Schwarz

$$= \int_a^b \frac{\partial}{\partial x}\underbrace{\frac{\partial F(x,t)}{\partial t}}_{=f(x,t)}dt$$

$$\Rightarrow \quad \boxed{\frac{d}{dx}\int_a^b f(x,t)\,dt = \int_a^b \frac{\partial f(x,t)}{\partial x}\,dt}$$

Theorem: This formula is true if $f(x,t)$ and $\frac{\partial f(x,t)}{\partial x}$ are continuous.

Ex.: $I(x) = \underbrace{\int_a^b \frac{\sin(xt)}{t}\,dt}_{\text{not so easy to integrate}}$, for $0<a<b$ (s.t. conditions of theorem hold)

$$\text{but } \frac{dI(x)}{dx} = \int_a^b \frac{\partial}{\partial x}\left(\frac{\sin(xt)}{t}\right)dt = \int_a^b \cos(tx)\,dt = \left.\frac{\sin(tx)}{x}\right|_a^b$$

This is often useful when functions depend on a parameter, e.g., temperature, density, heat capacity, initial values, ...

Advertisement: Please do some of the moodle exercises!

2.3 Critical Points

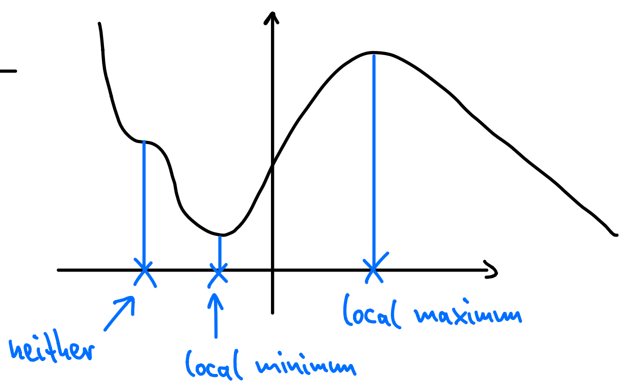
recall: let $f: \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable. Then x_0 is called a critical (or stationary) point if $\frac{df}{dx}(x_0) = 0$. Furthermore

↳ if $\frac{d^2f}{dx^2}(x_0) > 0$, x_0 is a local minimum

↳ if $\frac{d^2f}{dx^2}(x_0) < 0$, x_0 is a local maximum

↳ if $\frac{d^2f}{dx^2}(x_0) = 0$ inconclusive (can be max or min or neither)

Ex.:



What about max/min of $f(x_1, \ldots, x_n)$?

As in $\mathbb{R}$, we say that f has a local maximum (minimum) at $\vec{a}$ if $f(\vec{x}) \le f(\vec{a})$ $(f(\vec{x}) \ge f(\vec{a}))$ for all $\vec{x}$ in some neighborhood of $\vec{a}$.

Consider local max/min at $\vec{a}$:

↳ then for any $\vec{u} \in \mathbb{R}^n$, $|\vec{u}| = 1$, $g(t) = f(\vec{a} + t\vec{u})$ has a local max/min at $t = 0$

$$\Rightarrow 0 = \frac{dg}{dt}(0) = D_{\vec{u}} f(\vec{a}) = \left(\vec{\nabla} f(\vec{a})\right) \cdot \vec{u}$$

$$\Rightarrow \text{need } \vec{\nabla} f(\vec{a}) = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} =: \vec{0} \quad \text{, i.e., } \frac{\partial f}{\partial x_j}(\vec{a}) = 0 \quad \forall\, j = 1, \ldots, n$$

We have proven:

$\boxed{\begin{array}{l}\underline{\text{Theorem:}} \text{ Let } f:\mathbb{R}^n \to \mathbb{R} \text{ be differentiable at } \vec{a}. \text{ If } \vec{a} \text{ is a local max or min, then} \\ \qquad \vec{\nabla} f(\vec{a}) = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \quad (\text{i.e., } \vec{\nabla} f(\vec{a}) = \vec{0} \text{ is a necessary condition}).\end{array}}$

As before, $\boxed{\text{if } \vec{\nabla} f(\vec{a}) = \vec{0}, \text{ we call } \vec{a} \text{ a critical point.}}$

What can happen at $\vec{a}$ with $\nabla f(\vec{a}) = \vec{0}$?

<u>Ex.:</u> $f_1(x,y) = -x^2 - y^2$ ↳ local max

$f_2(x,y) = x^2 - y^2$ ↳ saddle point (max in one direction, min in another)

$f_3(x,y) = x^3 - y^2$ ↳ degenerate critical point

See geogebra.org pictures on next page.

As in $\mathbb{R}$, we need to look at second derivatives.

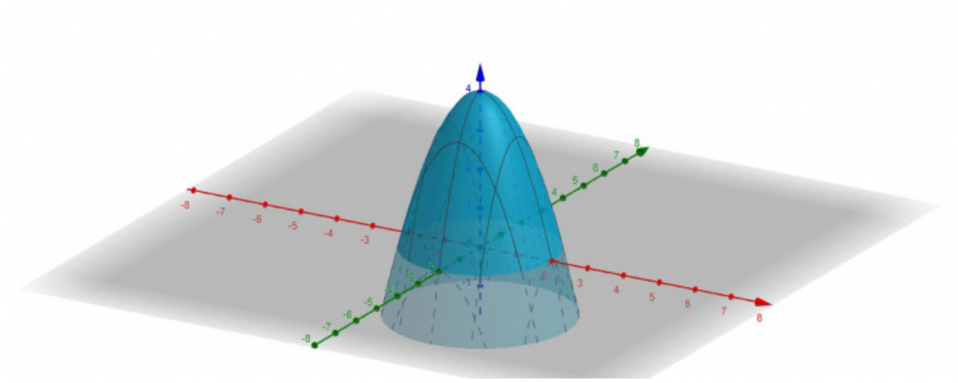
Taylor expansion: $f(\vec{a}+\vec{h}) = f(\vec{a}) + (\vec{\nabla} f)(\vec{a}) \cdot \vec{h} + \underbrace{\frac{1}{2} (\vec{h} \cdot \vec{\nabla})^2 f(\vec{a})} + R_{2,\vec{a}}(\vec{h})$

$$= \frac{1}{2} \left(\sum_{i=1}^n h_i \partial_{x_i} \right)^2 f(\vec{a}) = \frac{1}{2} \sum_{i,j=1}^n \left(\partial_{x_i} \partial_{x_j} f \right)(\vec{a})\, h_i h_j$$

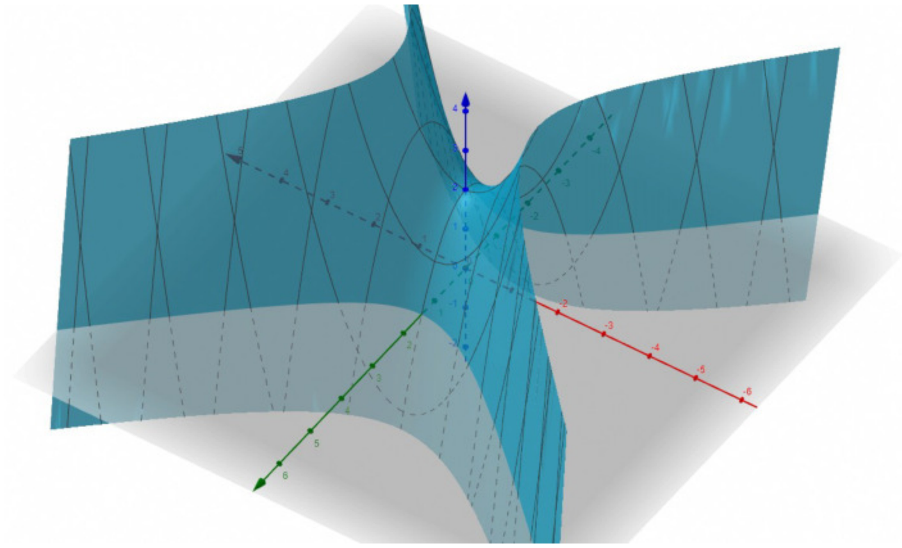
continue next time ...

Use https://www.geogebra.org/3d for generating the plots.

$f(x,y) = -x^2 - y^2 + 4$



$f(x,y) = x^2 - y^2 + 2$



$f(x,y) = x^3 - y^2 + 2$