Session 14
March 23, 2020

We continue the list of techniques for solving certain classes of ODEs from last time:

- $\boxed{\text{Second-order ODE } y''(x) = f(x, y'(x))}$ (but f does not depend on $y(x)$)

Here, we can introduce a new variable $p(x) = y'(x)$, i.e., $y''(x) = p'(x)$

Then we just need to solve the first-order ODE $\underbrace{p'(x)}_{y''(x)} = f(x, \underbrace{p(x)}_{y(x)})$ (e.g., if possible, with any of the techniques discussed before)

Once we found $p(x)$, we have $y(x) = \int p(x)dx + C$ (s.t. $y'(x) = p(x)$)

- $\boxed{\text{Second-order ODE } y''(x) = f(y(x), y'(x))}$ (but f does not explicitly depend on x; only implicitly through y and y')

Let us assume that $y' = p(y)$ for some function p (i.e., $y'(x) = p(y(x))$)

Then $y'' = (y')' = \frac{d}{dx}(p(y)) \overset{\text{chain rule}}{=} \frac{dp}{dy} \underbrace{\frac{dy}{dx}}_{p(y)} = p' p$

So we again have reduced the problem to solving the first-order ODE $\underbrace{p'(y)\,p(y) = f(y, y')}_{\text{just forget about } x \text{ here; solve this for } p \text{ as a function of } y}$

If we can solve this for $p(y)$, we finally still need to solve the first-order ODE $y'(x) = p(y(x))$ (autonomous).

One of the most important examples of a second-order ODE is the harmonic oscillator:

Newton's second law again: mass $\cdot$ acceleration $= m\frac{d^2x(t)}{dt^2} = \text{force} = -\underbrace{\overbrace{k}^{\text{some positive constant}}x}_{\text{force of a harmonic oscillator, e.g., a spring}}$

$\Rightarrow$ want to solve $m\frac{d^2x}{dt^2} = -kx$, or $\frac{d^2x}{dt^2} = -\omega^2 x$, $\omega := \sqrt{\frac{k}{m}}$

second method above: look at velocity $\frac{dx(t)}{dt} = v(x(t))$, then $\underbrace{\frac{dv}{dt}}_{\frac{d^2x(t)}{dt^2}} = \underbrace{\frac{dv}{dx}}_{v'(x)}\underbrace{\frac{dx}{dt}}_{v(x)} \overset{\text{Newton}}{=} -\omega^2 x$.

$\Rightarrow$ need to solve $\frac{dv(x)}{dx}v(x) = -\omega^2 x$

separation of variables: $v\,dv = -\omega^2 x\,dx \Rightarrow \underbrace{\int v\,dv}_{=\frac{1}{2}v^2} = -\omega^2 \underbrace{\int x\,dx}_{=\frac{1}{2}x^2}$

I chose $\frac{C_1}{2}$ instead of C_1 to make the notation easier

$\Rightarrow \frac{1}{2}v(x)^2 = -\frac{\omega^2}{2}x^2 + \frac{C_1}{2} \Rightarrow \underbrace{v(x) = \sqrt{C_1 - \omega^2 x^2}}$

next, we need to solve $\frac{dx(t)}{dt} = v(x(t)) = \sqrt{C_1 - \omega^2 x(t)^2}$

separation of variables again: $\frac{dx}{\sqrt{C_1 - \omega^2 x^2}} = dt$, so $\int \frac{1}{\sqrt{C_1 - \omega^2 x^2}}dx = \int dt$

This integral can be solved using the substitution $\omega x = \sqrt{C_1}\sin y$. Then $dx = \frac{\sqrt{C_1}}{\omega}\cos y\,dy$

and $\int \frac{1}{\sqrt{C_1 - \omega^2 x^2}}dx = \int \frac{1}{\sqrt{C_1}\underbrace{\sqrt{1-\sin^2 y}}_{1-\sin^2 y = \cos^2 y}} \frac{\sqrt{C_1}}{\omega}\cos y\,dy$

$$= \frac{1}{\omega}\int \frac{\cos y}{\sqrt{\cos^2 y}}dy$$

$$= \frac{1}{\omega}\int dy$$

$x = \frac{\sqrt{C_1}}{\omega}\sin y$

so $y = \arcsin\frac{\omega x}{\sqrt{C_1}}$

$$= \frac{y}{\omega}$$

(recall that arcsin is the inverse of sin, i.e., $\arcsin(\sin x) = x$)

$$= \frac{1}{\omega}\arcsin\frac{\omega x}{\sqrt{C_1}}$$

So integrating both sides in our equation above gives us $\frac{1}{\omega} \arcsin\left(\frac{\omega}{\sqrt{c_1}} x\right) = t + c_2$

$$\Rightarrow \quad x(t) = \frac{\sqrt{c_1}}{\omega} \sin(\omega t + \omega c_2)$$

or nicer: $x(t) = \underbrace{A_0}_{\text{amplitude}} \sin(\omega t + \underbrace{\varphi_0}_{\text{phase}})$

> Note: Often (e.g., in this example), one can try to guess the solution first. If it satisfies the ODE, and has enough constants, good!

We will come back to the topic of ODEs later in class, when we have more linear Algebra available.

Before we switch to linear Algebra, a few more remarks about how to easily understand some qualitative behavior of the solutions to ODEs.

3.3 Qualitative Properties of ODEs

Let us consider autonomous first order equations $\frac{dy}{dx} = v(y)$

By seperation of variables, we have $x = \int \frac{1}{v(y)} dy$ (if v is continuous and non-zero), but this integral might still be hard to solve.

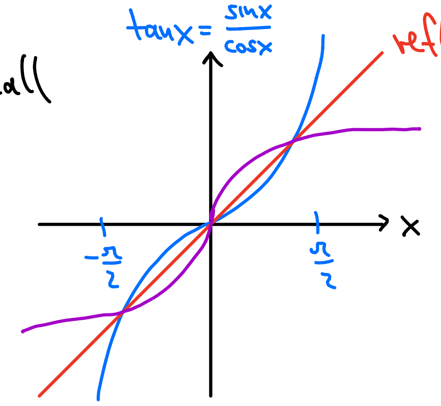
We would like to know: How does the solution $y(x)$ behave qualitatively for large x for different initial conditions?

Example: $v(y) = \arctan(y-\lambda_0)$

What is $\int \frac{1}{\arctan(y-\lambda_0)}\,dy$? Even Wolfram Alpha does not know...

But we know what v looks like:

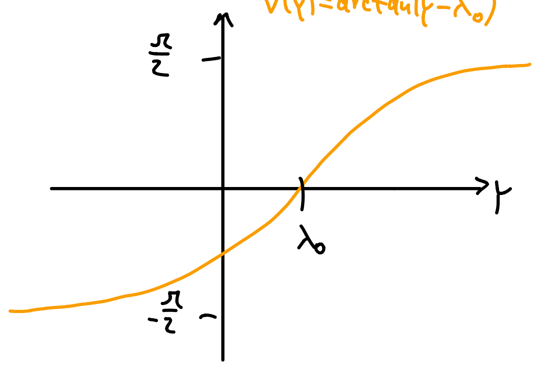
first, recall



reflection around $y=x$

gives us the inverse function

$\Longrightarrow$ (arctan shifted by λ_0)



Sometimes called slopes, or direction field

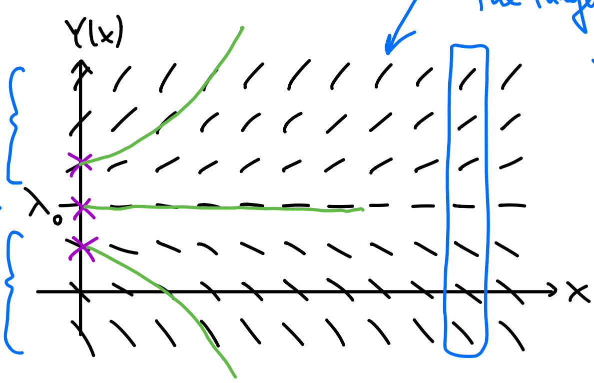
Now, we just draw the derivatives, i.e., little tangents at each point, since $\frac{dy}{dx} = v(y)$:

the tangents only depend on y here, not on x, since the ODE is autonomous

here, $v(y)$ is positive (and $v(y) \approx \frac{\pi}{2}$ for large y)

here, $v(y=\lambda_0) = 0$ $\longrightarrow$

here, $v(y)$ is negative (and $v(y) \approx -\frac{\pi}{2}$ for small y)



purple: different initial conditions at $x=0$

green: the qualitative solution: "we just follow the flow"

In particular, we can read off the following:

- if $y(0) = \lambda_0$, then $y(x) = \lambda_0 \;\; \forall x > 0$
- if $y(0) < \lambda_0$, then $y(x)$ decreases for all $x > 0$, and $y(x) \to -\infty$ for large x

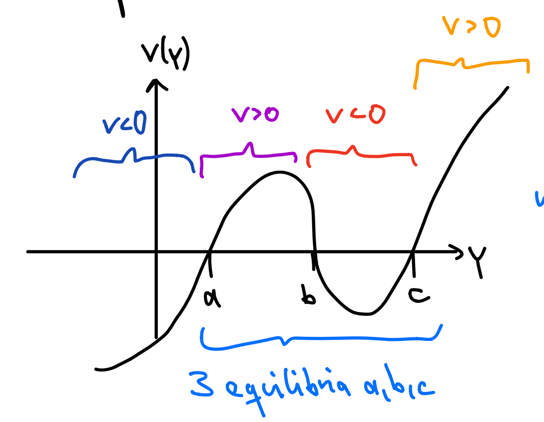
  ↳ even more, we find $y(x) \approx -\frac{\pi}{2} x$ for large x
- if $y(0) > \lambda_0$, then $y(x)$ increases for all $x > 0$, and $y(x) \to +\infty$ for large x

  ↳ even more, we find $y(x) \approx \frac{\pi}{2} x$ for large x

Note: The zeros of $v(y)$ are called **equilibrium positions**: if we start at a zero of v, i.e., y_0 s.t. $v(y_0) = 0$, then the solution will stay at y_0.

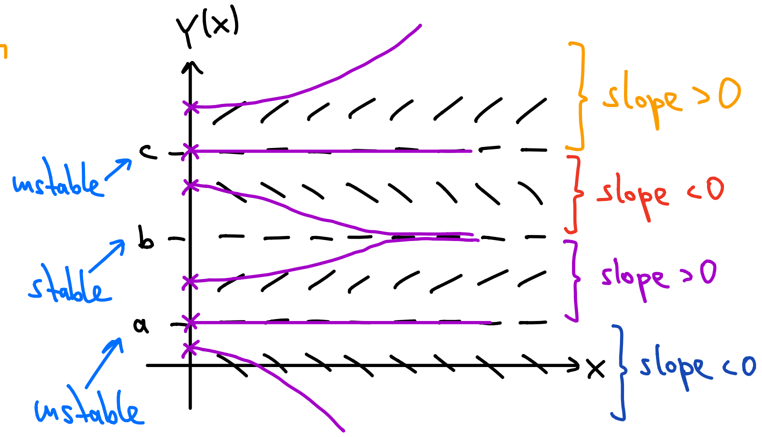
↳ if for small changes in the initial condition, the solution stays near the equilibrium, we call it **stable**, if the solution is lead away, we call it **unstable**

Examples:



Important: $v(y) = \frac{dy}{dx}$, so $v(y)$ gives us the slope of $y(x)$. We are not interested in the slope of $v(y)$!

Direction field:



Equilibria can be stable in one direction, but unstable in another: