4. Linear Algebra

Session 15
March 25, 2020

4.1 Review

Make sure to review the following concepts, results, and techniques from Calculus and Linear Algebra I:

- vector spaces in general, and in particular $\mathbb{R}^n$ as a vector space
- linear independence, span, basis, dimension

  linear independence: $\lambda_1 \vec{v}_1 + \ldots + \lambda_n \vec{v}_n = 0 \Rightarrow \lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$

  basis: linearly independent vectors that span the whole vector space

- inner product and norm (angles and lengths), in general, and in particular in $\mathbb{R}^n$

  $\vec{a}\cdot\vec{b} = \langle \vec{a}, \vec{b} \rangle = \sum_{i=1}^{n} a_i b_i = |\vec{a}|\,|\vec{b}| \cos\theta$, θ = angle between $\vec{a}, \vec{b}$

- matrix algebra, in particular multiplication of matrices and applying matrices to vectors

  $(AB)_{ik} = \sum_j A_{ij} B_{jk}$ $\qquad$ $(A\vec{v})_i = \sum_j A_{ij} v_j$

- matrices as linear maps/operators ($A(\vec{v} + \lambda\vec{w}) = A(\vec{v}) + \lambda A(\vec{w})$; $(A(\vec{e}_j))_i = A_{ij}$)

- homogeneous and inhomogeneous systems of linear equations, and their solutions with Gaussian elimination

- image and rank, nullspace/kernel and nullity, rank-nullity theorem

  image: $\{\vec{w} : \exists\, \vec{v} \text{ s.t. } A\vec{v} = \vec{w}\}$; rank: dim of this

  nullspace/kernel: $\{\vec{v} : A\vec{v} = 0\}$; nullity: dim. of this

  rank-nullity theorem: rank + nullity = dim. of vector space

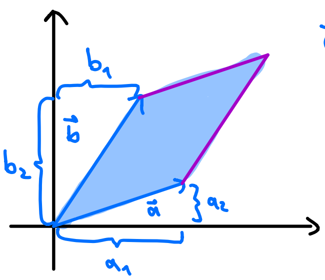
- matrix inverse (how to find A^{-1} such that $A^{-1}A = I$ = identity matrix)

Resources:
- Calculus and Linear Algebra I lecture notes from last semester
- Riley, Hobson, Bence book
- Leduc - Linear Algebra

4.2 Determinant

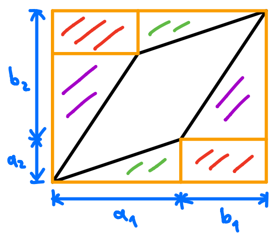
The determinant is an extremely useful tool in linear Algebra.

Let us ask the following question: What is the area of a parallelogram spanned by two vectors $\vec{a}$ and $\vec{b}$ in $\mathbb{R}^2$?



$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ (in the standard basis $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$)

Area =

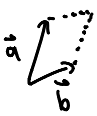


$$= (a_1+b_1)(a_2+b_2) - 2a_2b_1 - b_1b_2 - a_1a_2$$
$$= a_1a_2 + a_1b_2 + a_2b_1 + b_1b_2 - 2a_2b_1 - b_1b_2 - a_1a_2$$
$$= a_1b_2 - a_2b_1$$

There is one important subtlety: If we interchange $\vec{a}$ and $\vec{b}$, our expression for the area changes sign: $b_1a_2 - b_2a_1 = -(a_1b_2 - a_2b_1)$.

So area $= |a_1b_2 - a_2b_1|$. But the sign change is actually very useful, so we keep it in our definition below, and call the area oriented (

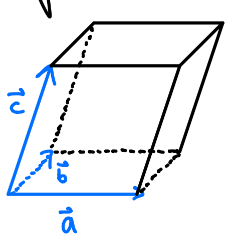
= positively oriented,



negatively oriented, just like the Right Hand Rule).

We call the oriented area of the parallelogram $\underbrace{\det}_{\text{determinant of } \binom{\vec{a}}{\vec{b}}}\underbrace{\begin{pmatrix} \vec{a} \\ \vec{b} \end{pmatrix}}_{\text{matrix with row vectors } \vec{a}, \vec{b}, \text{ i.e., } \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}} = a_1b_2 - a_2b_1$.

We could proceed analogously for higher dimensions, i.e., compute the volume of the parallelepiped in $\mathbb{R}^3$



(Side note: It turns out that the volume $= |(\vec{a} \times \vec{b}) \cdot \vec{c}|$.)

Instead we extract a general definition from the properties of our expression in $\mathbb{R}^2$.

Definition: A map $\det: \underbrace{\mathbb{R}^{n\times n}}_{\text{real } n\times n \text{ matrices}} \to \mathbb{R}$, $A \mapsto \det(A)$ (it takes in real $n\times n$ matrices and gives out a real number) is called determinant if it has the following properties:

D1) det is linear in each row, i.e.,

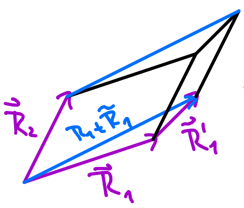
$$\det\begin{pmatrix} \vec{R}_1 + \lambda \vec{R}_1' \\ \vec{R}_2 \\ \vdots \\ \vec{R}_n \end{pmatrix} = \det\begin{pmatrix} \vec{R}_1 \\ \vec{R}_2 \\ \vdots \\ \vec{R}_n \end{pmatrix} + \lambda \det\begin{pmatrix} \vec{R}_1' \\ \vec{R}_2 \\ \vdots \\ \vec{R}_n \end{pmatrix},$$

where $\vec{R}_1', \vec{R}_1, \vec{R}_2, \dots, \vec{R}_n$ are row vectors, and $\lambda \in \mathbb{R}$, and analogous for all other rows.

= matrix with row vector $\vec{R}_1 + \lambda \vec{R}_1'$ in first row, and $\vec{R}_j$ in columns $2, \dots, n$

(For example: $\det\begin{pmatrix} a_1 + \lambda a_1' & a_2 + \lambda a_2' \\ b_1 & b_2 \end{pmatrix} = \det\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} + \lambda \det\begin{pmatrix} a_1' & a_2' \\ b_1 & b_2 \end{pmatrix}$.)

(In $\mathbb{R}^2$:



$\Rightarrow$ area of parallelogram spanned by $\vec{R}_1 + \vec{R}_1', \vec{R}_2$
$=$ (area spanned by $\vec{R}_1, \vec{R}_2$) + (area spanned by $\vec{R}_1', \vec{R}_2$).

Also: area of parallelogram spanned by $\lambda \vec{R}_1, \vec{R}_2 = \lambda \cdot$ (area spanned by $\vec{R}_1, \vec{R}_2$).)

D2) det changes sign if two rows are interchanged, i.e.,

$$\det\begin{pmatrix} \vec{R}_1 \\ \vec{R}_2 \\ \vec{R}_3 \\ \vdots \\ \vec{R}_n \end{pmatrix} = -\det\begin{pmatrix} \vec{R}_2 \\ \vec{R}_1 \\ \vec{R}_3 \\ \vdots \\ \vec{R}_n \end{pmatrix},$$

and analogous for any other pair of rows.

(In $\mathbb{R}^2$, we discussed this above: $\det\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = -\det\begin{pmatrix} b_1 & b_2 \\ a_1 & a_2 \end{pmatrix}$.)

D3) $\det(\underbrace{I}_{\text{identity matrix}}) = 1$ (det of identity matrix $= 1$)

$id = I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ (1's on the diagonal, all other entries 0)

(In $\mathbb{R}^2$: area of a unit square spanned by $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is equal to 1:

)

Note: We could have just as well used columns instead of rows in D1) and D2).

Important result (without proof): The properties D1), D2), D3) determine the map det <u>uniquely</u>.

Thus, we will always speak of <u>the</u> determinant.

Next, we will deduce further properties of det, which will also enable us to find a formula for general $n \times n$ matrices.

D4) The determinant of a matrix with two identical rows is zero.

Geometric reason: two identical vectors do not span a parallelogram, i.e., area is zero.

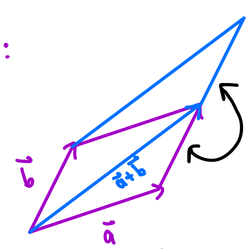
Proof: D2) says that

$$\det\begin{pmatrix} \vec{R} \\ \vec{R} \\ \vec{R}_3 \\ \vdots \\ \vec{R}_n \end{pmatrix} = -\underbrace{\det\begin{pmatrix} \vec{R} \\ \vec{R} \\ \vec{R}_3 \\ \vdots \\ \vec{R}_n \end{pmatrix}}_{\text{1st and 2nd row "interchanged"}}$$

(and $x = -x$ implies $x = 0$)

D5) Adding a multiple of one row to another row leaves the determinant unchanged.

Geometric reason: one has to visualize this:



same triangles, so area unchanged.

Proof:
$$\det\begin{pmatrix} \vec{R}_1 + \lambda\vec{R}_2 \\ \vec{R}_2 \\ \vdots \\ \vec{R}_n \end{pmatrix} \overset{D1)}{=} \det\begin{pmatrix} \vec{R}_1 \\ \vec{R}_2 \\ \vdots \\ \vec{R}_n \end{pmatrix} + \underbrace{\lambda \det\begin{pmatrix} \vec{R}_2 \\ \vec{R}_2 \\ \vec{R}_3 \\ \vdots \\ \vec{R}_n \end{pmatrix}}_{=0 \text{ according to D4)}}.$$

Note: this is very good, just think of Gaussian elimination.

D7) If a matrix has one row with only 0's, the determinant is zero.

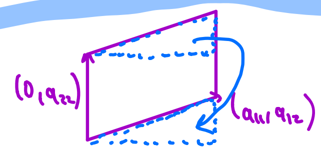
numbering according to Ledvc book

Geometric reason: clear, since then area $= 0$.

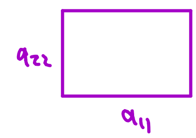
Proof: $\det\begin{pmatrix}\vec{0}\\ \vec{R}_2\\ \vdots\\ \vec{R}_n\end{pmatrix} = \det\begin{pmatrix}2\cdot\vec{0}\\ \vec{R}_2\\ \vdots\\ \vec{R}_n\end{pmatrix} \overset{D1)}{=} 2\det\begin{pmatrix}\vec{0}\\ \vec{R}_2\\ \vdots\\ \vec{R}_n\end{pmatrix}$ (and $x = 2x$ implies $x = 0$)

D6) The determinant of an upper triangular matrix is equal to the product of the diagonal entries.

Geometric reason: $\det\begin{pmatrix}a_{11} & a_{12}\\ 0 & a_{22}\end{pmatrix} =$ area of



$=$ area of



$= a_{11}\cdot a_{22}$.

Proof: consider matrix $A = \begin{pmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ & a_{22} & \cdots & a_{2n}\\ & 0 & \ddots & \vdots\\ & & & a_{nn}\end{pmatrix}$

all entries 0 in lower triangle

now do backwards induction: start with last two rows: $\begin{matrix}0 & \cdots & 0 & a_{n-1,n-1} & a_{n-1,n}\\ 0 & & \cdots & 0 & a_{nn}\end{matrix}$

If $a_{nn} = 0 \Rightarrow$ done due to D7)

Otherwise add $-\frac{a_{n-1,n}}{a_{nn}}$ times last row to second-last row. This does not change the determinant according to D5). The result is $\begin{matrix}0 \cdots 0 & a_{n-1,n-1} & 0\\ 0 \cdots 0 & 0 & a_{nn}\end{matrix}$.

Continue this until matrix is diagonal, i.e., $\begin{pmatrix}a_{11} & & 0\\ & a_{22} & \\ 0 & & \ddots & \\ & & & a_{nn}\end{pmatrix}$.

Finally, $\det\begin{pmatrix}a_{11} & & 0\\ & a_{22} & \\ 0 & & \ddots \\ & & & a_{nn}\end{pmatrix} \overset{D1)}{=} a_{11}\det\begin{pmatrix}1 & & 0\\ & a_{22} & \\ 0 & & \ddots & \\ & & & a_{nn}\end{pmatrix} \overset{D1)}{=} a_{11}a_{22}\det\begin{pmatrix}1 & & 0\\ & 1 & \\ 0 & & a_{33} & \\ & & & \ddots \\ & & & & a_{nn}\end{pmatrix}$

$$= \ldots = a_{11}a_{22}\cdots a_{nn}\underbrace{\det\begin{pmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{pmatrix}}_{=1 \text{ according to D3)}} = \prod_{i=1}^{n} a_{ii}.$$

Technique 1 to compute determinants: Use D5) to bring matrix in upper triangular form, and then D6) to compute determinant.

Example: $\det\begin{pmatrix} 1 & 4 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \underset{\substack{\text{add } (-1)\cdot \text{first row} \\ \text{to second row}}}{=} \det\begin{pmatrix} 1 & 4 & 3 \\ 0 & -3 & -1 \\ 0 & 1 & 2 \end{pmatrix} \underset{\substack{\text{add } \frac{1}{3}\cdot \text{second row} \\ \text{to third row}}}{=} \det\begin{pmatrix} 1 & 4 & 3 \\ 0 & -3 & -1 \\ 0 & 0 & \frac{5}{3} \end{pmatrix} \underset{D6)}{=} 1\cdot(-3)\cdot\frac{5}{3} = -5.$