Moodle Exercise Set 4

Calculus and Linear Algebra II

Spring 2020

1. Let $f(x, y) = 1 + 2x\sqrt{y}$. Find the directional derivative at $(3, 4)$ in the direction $\vec{v} = (4, -3)$.
2. Let $f(x, y, z) = xe^y + ye^z + ze^x$. Find the directional derivative at $(0, 0, 0)$ in the direction $\vec{v} = (5, 1, 2)$.
3. Let $f(x, y, z) = (x + 2y + 3z)^{3/2}$. Find the directional derivative at $(1, 1, 2)$ in the direction $\vec{v} = (0, 2, -1)$.
4. Let $f(x, y) = \sqrt{xy}$. Find the directional derivative at $(2, 8)$ in the direction of the point $(5, 4)$.
5. Consider $f(x, y) = \sin(xy)$ at the point $(1, 0)$. In which direction does f have the maximum rate of change?
6. Determine whether the differential $F = Pdx + Qdy = e^x \sin(y)dx + e^x \cos(y)dx$ is exact. If the differential is inexact what is the difference $\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}$?
7. Determine whether the differential $F = Pdx + Qdy = (ye^x \sin(y))dx + (e^x + x\cos(y))dy$ is exact. If the differential is inexact what is the difference $\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}$?
8. A force $F = (F_x, F_y)$ written as a differential $F = F_x dx + F_y dy$ is called conservative if there exists V such that $F = -dV$. When an object moves in a closed loop (i.e., the motion has the same start and end points) in a conservative force field, the net work done by the force is zero. Consider a planet moving around the gravitational influence of a star. What is the work done by the force field

$$F = \frac{kx}{x^2 + y^2} dx + \frac{ky}{x^2 + y^2}$$

when the planet finishes one revolution? Above, k is a constant.
9. Let $f(x, y) = e^{x^2 + y}$. Find the second order Taylor polynomial that approximates f at $(0, 0)$.
10. Let $f(x, y) = x\sin(x - y)$. Find the second order Taylor polynomial that approximates f at $(1, 1)$.
11. Let f be a differentiable function of x and y and $g(u, v) = f(e^u + \sin(v), e^u + \cos(v))$. Use the table of values below to find $\partial_u g(0, 0)$ and $\partial_v g(0, 0)$.

	f	g	$\partial_x f$	$\partial_y f$
$(0, 0)$	3	6	4	8
$(1, 2)$	6	3	2	5

12. Let

$$f(x, y) = \frac{xy}{x^2 + y^2}.$$

(Observe the similarity to the function from the bonus problem in Homework 2). First use the change of coordinates $x = r\cos(\theta)$ and $y = r\sin(\theta)$ and then compute $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial \theta}$ when $(x, y) \neq 0$.
13. A spherical pendulum consists of a mass m attached to a string of length l fixed at the origin (in $\mathbb{R}^3$). The motion of the pendulum can be described by Cartesian coordinates $(x(t), y(t), z(t))$. The potential energy V and the kinetic energy T are given by

$$V = mgz(t),$$
$$T = \frac{m}{2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right]$$

where g is a constant. What is the difference $L = T - V$ of the kinetic energy and the potential energy in spherical coordinates? The coordinate transform is given by

$$x = l\sin(\theta)\cos(\varphi),$$
$$y = l\sin(\theta)\sin(\varphi),$$
$$z = -l\cos(\theta).$$

Above l is the distance of the mass from the origin (assume that $\frac{dl}{dt} = 0$) and θ is the angle with respect to the z-axis and φ the angle (on the xy-plane) with respect to the x-axis.

14. Compute the derivative $\frac{d}{dx}\int_0^1 (2t + x^3)^2 dt$.

15. Compute the derivative $\frac{d}{dx}\int_0^1 \frac{t^x - e^t}{\ln(t)} dt$ when $x > 1$.