Jacobs University February 17, 2020
Spring 2020

Foundations of Mathematical Physics

Homework 2

Due on February 24, 2020

Problem 1 [6 points]: Bijectivity of the Fourier Transform
Finish the proof that was sketched in class, that the Fourier transform $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is a continuous bijection with continuous inverse $\mathcal{F}^{-1}$ (with $\mathcal{F}^{-1}$ as defined in class).

Problem 2 [2 points]: Plancherel on $\mathcal{S}$
Prove that for $f, g \in \mathcal{S}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \widehat{f}(x) g(x) dx = \int_{\mathbb{R}^d} f(x) \widehat{g}(x) dx,$$

and in particular,

$$\int_{\mathbb{R}^d} |\widehat{f}(k)|^2 dk = \int_{\mathbb{R}^d} |f(x)|^2 dx.$$

Problem 3 [6 points]: Dilations
Let $p \in [1, \infty)$, $\sigma > 0$ and $D^p_\sigma : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d), f(x) \mapsto (D^p_\sigma f)(x) = \sigma^{-d/p} f(x/\sigma)$ the L^p dilation with σ.

(a) Show that $D^p_\sigma : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ is continuous and $\|D^p_\sigma f\|_{L^p(\mathbb{R}^d)} = \|f\|_{L^p(\mathbb{R}^d)}$.

(b) Compute $\mathcal{F} D^2_\sigma f$ and interpret the result.

Problem 4 [6 points]: Convolution
Let $f \in C(\mathbb{R}^d)$ be bounded and let $\varphi \in \mathcal{S}(\mathbb{R}^d)$ with $\varphi \geq 0$ and $\|\varphi\|_{L^1(\mathbb{R}^d)} = 1$. Define $f_\sigma := f * (D^1_\sigma \varphi)$, where the D^1_σ are the dilations from Problem 3.

(a) Show that $f_\sigma \in C^\infty(\mathbb{R}^d)$.

(b) Show that $f_\sigma(x) \to f(x)$ as $\sigma \to 0$ for all $x \in \mathbb{R}^d$.

(c) Show that the convergence in b) is uniform on each compact interval. You may additionally assume $\varphi \in C^\infty_0(\mathbb{R}^d)$.