Jacobs University
Spring 2020

February 24, 2020

# Foundations of Mathematical Physics

## Homework 3

Due on March 2, 2020

**Problem 1 [3 points]: Derivatives of the Step Function**
Let $\Theta : \mathbb{R} \to \mathbb{R}$ be the Heaviside step function, i.e.,

$$\Theta(x) = \begin{cases} 1 & \text{, for } x \geq 0 \\ 0 & \text{, for } x < 0, \end{cases}$$

and $T_\Theta$ the corresponding distribution. Compute all distributional derivatives of $T_\Theta$ (i.e., the derivative to arbitrary order).

**Problem 2 [3 points]: Dilations ctd.**
We continue Problem 3 from Homework 2. How does one have to define $\tilde{D}^p_\sigma : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ in order to extend $D^p_\sigma$?

**Problem 3 [8 points]: Convolution in $L^p$**
Let $1 \leq p < \infty$ and $f \in L^p(\mathbb{R}^d)$.

(a) Using the Hölder inequality

$$\|fg\|_{L^1} \leq \|f\|_{L^p}\|g\|_{L^q},$$

where $1/p + 1/q = 1$, show that for $g \in L^1(\mathbb{R}^d)$ we have

$$\|f * g\|_{L^p} \leq \|f\|_{L^p}\|g\|_{L^1}.$$

*Hint: Show first that $f * g \in L^p(\mathbb{R}^d)$ by using $L^p(\mathbb{R}^d) = (L^q(\mathbb{R}^d))'$ (dual space of $L^q$, $1/p + 1/q = 1$). The inequality can then be shown to follow from this consequence of the Hahn-Banach theorem: For all $h \in L^p(\mathbb{R}^d)$ there is an $\tilde{h} \in L^q(\mathbb{R}^d)$ with $\|\tilde{h}\|_{L^q} = 1$ and*

$$\|h\|_{L^p} = \tilde{h}(h) := \int_{\mathbb{R}^d} \tilde{h}(x)h(x)dx.$$

(b) Let $\varphi \in C_0^\infty(\mathbb{R}^d)$ with $\varphi \geq 0$ and $\int_{\mathbb{R}^d} \varphi = 1$. Define $f_\sigma := f * (D^1_\sigma \varphi)$ as in Problem 3 from Homework 2. Using (a), show that $f_\sigma$ converges to $f$ in $L^p(\mathbb{R}^d)$ as $\sigma \to 0$, i.e.,

$$\lim_{\sigma \to 0} \|f_\sigma - f\|_{L^p} = 0.$$

*Hint: Use that $C_0(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ and Problem 4 from Homework 2.*

**Problem 4 [6 points]: Multiplication Operators on $L^p$**
Let $V : \mathbb{R}^d \to \mathbb{R}$ be measurable and $1 \leq p \leq \infty$. Show that $V$ defines a continuous multiplication operator

$$M_V : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d), \psi \mapsto V\psi$$

if and only if $V \in L^\infty(\mathbb{R}^d)$. Show that then

$$\|M_V\|_{\mathcal{L}(L^p)} := \sup_{\|f\|_{L^p}=1} \|M_V f\|_{L^p} = \|V\|_\infty.$$

*Hint: It might be easier to do this problem for continuous $V : \mathbb{R}^d \to \mathbb{R}$. Such a $V$ is in $L^\infty$ if and only if it is bounded.*