Jacobs University
Spring 2020

March 23, 2020

Foundations of Mathematical Physics

Homework 6

Due on March 30, 2020

Problem 1 [6 points]: Sequences of operators

Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space with an orthonormal basis $(\varphi_n)_n$. Give one example (with proof, for (a) and (b) separately), of a sequence $(A_n)_n$ in $\mathcal{L}(\mathcal{H})$ and an A in $\mathcal{L}(\mathcal{H})$, such that

(a) A_n converges weakly to A but not strongly;

(b) A_n converges strongly to A but not in norm.

Problem 2 [8 points]: Fourier transform

(a) Let $f \in C_c^\infty(\mathbb{R}^d)$, and let $0 < \alpha < d$. Prove that then

$$c_\alpha \mathcal{F}^{-1}(|k|^{-\alpha} \widehat{f}(k))(x) = c_{d-\alpha} \int |x-y|^{\alpha-d} f(y) \, \mathrm{d}y$$

for some constant c_α, and determine c_α explicitly. *Hint:* $\int_0^\infty e^{-\pi k^2 \lambda} \lambda^{\alpha/2-1} \, \mathrm{d}\lambda$.

Note: In this sense we can give a meaning to the Fourier transform of $|x|^{\alpha-d}$.

(b) The function $g(x) = |x|^{\alpha-d}$ is not in $L^1(\mathbb{R}^d)$. Other than above, how and why exactly can we define the Fourier transform of g?

Problem 3 [6 points]: Sobolev Lemma

For $m \in \mathbb{Z}$, we define the m-th Sobolev space $H^m(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$ as the set of $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $\widehat{f}$ is measurable and $(1+|k|^2)^{m/2} \widehat{f} \in L^2(\mathbb{R}^d)$. Now let $\ell \in \mathbb{N}_0$ and $f \in H^m(\mathbb{R}^d)$ with $m > \ell + \frac{d}{2}$. Prove that then $f \in C^\ell(\mathbb{R}^d)$ and $\partial^\alpha f \in C_\infty(\mathbb{R}^d)$ for all $|\alpha| \leq \ell$. *Hint: Riemann-Lebesgue and Cauchy-Schwarz.*