Jacobs University — April 6, 2020
Spring 2020

Foundations of Mathematical Physics

Homework 7

Due on April 20, 2020

Problem 1 [6 points]: Properties of generators
Prove the following properties of a generator H of a strongly continuous unitary group $U(t)$:

(a) $U(t)D(H) = D(H)$ for all $t \in \mathbb{R}$, i.e., the domain $D(H)$ is invariant under $U(t)$,

(b) $\big[H, U(t)\big]\psi := HU(t)\psi - U(t)H\psi = 0$ for all $\psi \in D(H)$,

(c) H is symmetric, i.e., $\langle H\psi, \varphi\rangle = \langle \psi, H\varphi\rangle$ for all $\psi, \varphi \in D(H)$,

(d) U is uniquely determined by H and H is uniquely determined by U.

Problem 2 [7 points]: Group of translations

(a) Let $a \in \mathbb{R}$ and $(T_a(t))_{t\in\mathbb{R}}$ the group of translations on $L^2(\mathbb{R})$, i.e., $(T_a(t)\psi)(x) := \psi(x-at)$.

 (i) Prove that the generator of T_a is $-ia\frac{d}{dx}$ with domain $H^1(\mathbb{R})$.

 (ii) Prove that $T_a(t)$ converges weakly, but not strongly to 0 on $L^2(\mathbb{R})$ for $t \to \infty$.

(b) Define $H^m(\mathbb{S}^1)$ in analogy to $H^m(\mathbb{R})$ by regarding $\mathbb{S}^1$ as the interval $[0, 2\pi)$ with identifying 0 and 2π, and by using Fourier series instead of Fourier transform. Find the unitary group generated by $-ia\frac{\mathrm{d}}{\mathrm{d}\varphi}$ with domain $H^1(\mathbb{S}^1)$ and $a \in \mathbb{R}$.

Problem 3 [7 points]: Multiplication operators as generators
Let $V : \mathbb{R}^d \to \mathbb{R}$ be measurable and let $M_V : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ be the multiplication operator from Problem 4 in Homework sheet 3.

(a) Prove that M_V with domain $D(M_V) := \{\psi \in L^2(\mathbb{R}^d) : V\psi \in L^2(\mathbb{R}^d)\}$ generates a unitary group $U(t) = e^{-iVt}$ if $D(M_V)$ is dense in $L^2(\mathbb{R}^d)$. *Hint: The group property and strong continuity are easy, so the conditions from Def. 3.31 have to be checked. Note that the second condition from this definition needs to be checked only at $t = 0$ due to the group property and is easily established once the first condition is proven.*

(b) Prove that $(U(t))_{t\in\mathbb{R}}$ is norm continuous if and only if $V \in L^\infty(\mathbb{R}^d)$.