Jacobs University
Spring 2020

April 27, 2020

Foundations of Mathematical Physics

Homework 9

Due on May 4, 2020

Problem 1 [8 points]: Closable and non-closable operators

(a) Let T be closable (i.e., T has a closed extension). Prove that $\overline{\Gamma(T)}$ is the graph of a linear operator $\overline{T}$.

(b) Let T be symmetric (i.e., as shown in class, T is in particular closable). Prove that $\overline{T}$ is also symmetric.

(c) Prove that the Dirac-distribution $(\delta, C_0(\mathbb{R}))$ as an unbounded operator from $L^2(\mathbb{R})$ to $\mathbb{C}$ is not closable. Determine $\overline{\Gamma(\delta)}$. (Recall that $C_0(\mathbb{R})$ denotes the continuous functions on $\mathbb{R}$ with compact support.)

Problem 2 [4 points]: Self-adjointness

Let $U : \mathcal{H}_1 \to \mathcal{H}_2$ be unitary and $(H, D(H))$ self-adjoint on $\mathcal{H}_1$. Prove that $(UHU^*, UD(H))$ is self-adjoint on $\mathcal{H}_2$.

Problem 3 [8 points]: Polarization

(a) Let X be a complex vector space and $B : X \times X \to \mathbb{C}$ a sesquilinear form, i.e., B is antilinear in the first, and linear in the second argument. Prove that for all $x, y \in X$ we have

$$B(x,y) = \frac{1}{4}\Big(B(x+y,x+y) - B(x-y,x-y) - iB(x+iy,x+iy) + iB(x-iy,x-iy)\Big).$$

(b) Let $\mathcal{H}$ be a Hilbert space. Using (a), prove that a densly defined linear operator $(T, D(T))$ is symmetric on $\mathcal{H}$ if and only if

$$\big\langle \psi, T\psi \big\rangle \in \mathbb{R} \quad \text{for all } \psi \in D(T).$$

(c) Let C be an antilinear isometry. Prove that

$$\big\langle C\psi, C\varphi \big\rangle = \big\langle \varphi, \psi \big\rangle \quad \text{for all } \psi, \varphi \in \mathcal{H}.$$