Session 3
Feb. 10, 2020

- fact: for $d=3$, wave function is either
  - symmetric, i.e., $\psi(\dots,x_j,\dots,x_k,\dots) = \psi(\dots,x_k,\dots,x_j,\dots)$, then it describes bosons $\quad (\psi(x_1,\dots,x_N) = \psi(x_{\sigma(1)},\dots,x_{\sigma(N)}) \ \forall \sigma \in S_N)$
  - or antisymmetric, i.e., $\psi(\dots,x_j,\dots,x_k,\dots) = -\psi(\dots,x_k,\dots,x_j,\dots)$, then it describes fermions $\quad (\psi(x_1,\dots,x_N) = (-1)^\sigma \psi(x_{\sigma(1)},\dots,x_{\sigma(N)}) \ \forall \sigma \in S_N)$

↳ very different physics: – bosons: Bose-Einstein-Condensation, ...
– fermions: Fermi pressure, superconductivity, ...
$\Rightarrow$ active research topics

note: • reason for symm./antisymm. is that really ${}^N\mathbb{R}^3 := \{q \subset \mathbb{R}^3 : |q| = N\}$ is the right configuration space, and not $\mathbb{R}^{3N}$ ("all particles are indistinguishable"), for the same particle species; also, we need
$$|\psi(x_1,\dots,x_N)|^2 = |\psi(x_{\sigma(1)},\dots,x_{\sigma(N)})|^2$$

• ${}^N\mathbb{R}^3$ is still simply connected, which leads to the boson-fermion alternative
• ${}^N\mathbb{R}^2$ is not simply connected $\Rightarrow$ many more possibilities, so-called anyons! (relevant for quasi 2 dim. materials)

- time-independent Schrödinger equation: eigenvalue problem $\boxed{H\phi_E = E\phi_E}$

E = eigenvalues, energies
ϕ_E = eigenstates

↳ give solution $\psi(t) = e^{-iEt}\phi_E$ to time-dependent Schrödinger equation

↳ $E_0 = \inf\{E\} = \inf\limits_{\Phi} \langle \Phi, H\Phi \rangle$, if it exists, is called ground state energy

if a minimizer Φ_{E_0} exists, it is called ground state

very relevant, since matter tends to radiate until it reaches lowest energy state

↳ Φ_E with $E > E_0$ are called excited states

↳ active research topics: find/approximate: • E_0, Φ_{E_0}

• low lying E, Φ_E

(• all E, Φ_E rarely possible)

• Hamiltonian of non-relativistic matter for N electrons and M nuclei at positions $Y_1(t), \dots, Y_M(t)$ (Born-Oppenheimer approximation), with charges $\underbrace{Z_1, \dots, Z_K}$: in multiples of the elementary charge e

$$\boxed{H = \sum_{j=1}^{N} \frac{\hbar^2}{2m_e}(-\Delta_{x_j}) + \hbar c \alpha \left(\underbrace{\sum_{1 \le j < k \le M} \frac{Z_j Z_k}{|Y_j(t) - Y_k(t)|}}_{=\text{const} = \text{energy of nuclei}} - \underbrace{\sum_{j=1}^{N} \sum_{k=1}^{M} \frac{Z_k}{|x_j - Y_k(t)|}}_{=\text{attractive external field of nuclei}} + \underbrace{\sum_{1 \le j < k \le N} \frac{1}{|x_j - x_k|}}_{=\text{repulsive Coulomb interaction of electrons}} \right)}$$

↳ m_e = electron mass

↳ neutral molecule: $\sum_{j=1}^{M} Z_j = N$

This Hamiltonian describes – sometimes with small modifications – non relativistic matter (e.g., chemistry, conductivity, ...)

Central topic of this class:

For which $\psi(t=0)$ and V does Schrödinger equation have global solutions, and in which sense?

general idea: regard Schrödinger equation as an ODE $i\frac{d}{dt}\psi(t) = H\psi(t)$ for

$\psi : \mathbb{R} \to \mathcal{H}$ = some Hilbert space, usually $L^2(\mathbb{R}^{dN})$

difficulty: • $\mathcal{H}$ infinite dimensional

• H unbounded