Session 4
Feb. 12, 2020

2. The Free Schrödinger Equation

free SE: $V=0$, i.e., $i\partial_t \psi(t,x) = -\frac{1}{2}\Delta_x \psi(t,x)$, $\psi: \mathbb{R}_t \times \mathbb{R}_x^d \to \mathbb{C}$

recall: solutions to the stationary SE $-\frac{1}{2}\Delta_x \phi(x) = E\,\phi(x)$ give us solutions

$$\psi(t,x) = e^{-iEt}\phi(x)$$

formally, the "eigenfunctions" of $-\frac{1}{2}\Delta_x$ are plane waves

$\boxed{\phi_k(x) = e^{ik\cdot x}} = e^{i\sum_{j=1}^{d} k_j x_j}$, for any $k \in \mathbb{R}^d$ $\left(\text{since } -\frac{1}{2}\Delta_x \phi_k(x) = \frac{1}{2}k^2 \phi_k(x)\right)$

$\Rightarrow$ this gives solutions $\psi_k(t,x) = e^{-i\frac{k^2}{2}t}e^{ikx}$ of the free SE

but $|\psi_k(t,x)|^2 = 1$, so on $\mathbb{R}^d$: $\int_{\mathbb{R}^d} |\psi_k(t,x)|^2 dx = \infty$, but we want $\int_{\mathbb{R}^d} |\psi|^2 = 1$

by linearity, we find that formally $\psi(t,x) = \int f(k)\,\psi_k(t,x)\,dx = \int f(k)\,e^{-i\frac{k^2}{2}t}e^{ikx}\,dx$

is also a solution, and $f: \mathbb{R}^d \to \mathbb{C}$ is determined by the initial condition:

$$\psi(0,x) = \int f(k)\,e^{ikx}\,dx$$

Conclusion: we need to study Fourier transform on $\mathbb{R}^d$

2.1 Fourier Transform on $\mathcal{S}$

recall: $\boxed{L^p(\mathbb{R}^d) := \left\{ f: \mathbb{R}^d \to \mathbb{C} : \|f\|_p = \left(\int |f|^p\right)^{\frac{1}{p}} < \infty \right\}}$

note: • all integrals are Lebesgue integrals here

also called ess sup f, the essential supremum

• $L^\infty(\mathbb{R}^d) := \left\{ f: \mathbb{R}^d \to \mathbb{C} : \|f\|_\infty := \inf\{C \geq 0 : |f(x)| \leq C \text{ for almost all } x\} < \infty \right\}$

note: one can show that $\lim\limits_{p\to\infty} \|f\|_p = \|f\|_\infty \quad (\forall f \in L^\infty \cap L^q \text{ for some } q)$

• for all $1 \leq p \leq \infty$, $L^p(\mathbb{R}^d)$ are Banach spaces (i.e., complete normed vector spaces) if one identifies functions that agree almost everywhere (always assumed)

$\Rightarrow$ really, $L^p(\mathbb{R}^d)$ are vector spaces of equivalence classes of functions

• only $L^2(\mathbb{R}^d)$ is a Hilbert space with scalar product $\langle f, g\rangle = \int \bar{f} g$

Def. 2.1: Let $f \in L^1(\mathbb{R}^d)$, then we define the

• Fourier transform of f as $\boxed{\hat{f}(k) = (\mathcal{F}f)(k) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(x) e^{-ikx}\, dx}$

• inverse Fourier transform of f as $\boxed{\check{f}(x) = (\mathcal{F}^{-1}f)(x) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(k) e^{ikx}\, dk}$

Next: want to know about regularity of $\hat{f}$ (i.e., continuity, differentiability)

$\rightarrow$ need to take derivative of integral with parameter

Lemma 2.2: Integrals with Parameter

Let $I(\gamma) = \int_{\mathbb{R}^d} f(x,\gamma)\,dx$, with $f: \mathbb{R}^d \times \Gamma \to \mathbb{C}$, where $\Gamma \subset \mathbb{R}$ an open interval, and let $f(x,\gamma) \in L^1(\mathbb{R}^d)$ for all fixed $\gamma \in \Gamma$.

a) If $\gamma \mapsto f(x,\gamma)$ is continuous for $\overbrace{\text{almost all}}^{:=\text{a.a.}}$ $x \in \mathbb{R}^d$

and if $\exists\, g \in L^1(\mathbb{R}^d)$ with $\sup_{\gamma\in\Gamma} |f(x,\gamma)| \leq g(x)$ for a.a. $x \in \mathbb{R}^d$,

then $I(\gamma)$ is continuous.

b) If $\gamma \mapsto f(x,\gamma)$ is continuously differentiable $\forall x \in \mathbb{R}^d$

and if $\exists\, g \in L^1(\mathbb{R}^d)$ with $\sup_{\gamma\in\Gamma} |\partial_\gamma f(x,\gamma)| \leq g(x)$ $\forall x \in \mathbb{R}$,

then $I(\gamma)$ is continuously differentiable and

$$\frac{d\,I(\gamma)}{d\gamma} = \frac{d}{d\gamma}\int_{\mathbb{R}^d} f(x,\gamma)\,dx = \int_{\mathbb{R}^d} \partial_\gamma f(x,\gamma)\,dx.$$

Proof: HW. Use dominated convergence for the Lebesgue integral □

(Note: Lemmas like this one are one of the main advantages of Lebesgue over Riemann integral.)

next: a basic property of the Fourier transform on L^1