Session 8
Feb. 26, 2020

last time: • pseudodifferential operators $f(-i\nabla) = (\mathcal{F}^{-1} f(k) \mathcal{F} \psi)(x)$
defined via multiplication in Fourier space
• convolution: $(f * g)(x) = \int f(x-y) g(y)\, dy$

Lemma 2.23: For $f, g, h \in \mathcal{S}$ we have

a) $(f*g)*h = f*(g*h)$ and $f*g = g*f$

b) the map $\mathcal{S} \to \mathcal{S}$, $g \mapsto f*g$ is continuous

c) $\widehat{f*g} = (2\pi)^{\frac{d}{2}} \hat{f} \cdot \hat{g}$ and $\widehat{f g} = (2\pi)^{-\frac{d}{2}} \hat{f} * \hat{g}$,

in particular $g(-i\nabla) f = \mathcal{F}^{-1} M_g \mathcal{F} f = \mathcal{F}^{-1} g \hat{f} = (2\pi)^{-\frac{d}{2}} \check{g} * f$

Proof: • a) and c) are direct calculations
• then b) follows since $f * g = (2\pi)^{\frac{d}{2}} \mathcal{F}^{-1} \hat{f} \hat{g}$, i.e., composition of continuous maps $\square$

Example: heat equation: $f(t,x) = W(t, -i\nabla) f(0,x)$, $W(t,k) = e^{-\frac{k^2}{2} t}$

$$= (2\pi)^{-\frac{d}{2}} \left((\mathcal{F}^{-1} W_t) * f_0 \right)(x)$$

with heat kernel $G(t,x) = (2\pi)^{-\frac{d}{2}} (\mathcal{F}^{-1} W)(t,x) = (2\pi t)^{-\frac{d}{2}} e^{-\frac{x^2}{2t}}$ we find

$$f(t,x) = \left(G(t) * f_0 \right)(x) = (2\pi t)^{-\frac{d}{2}} \int e^{-\frac{(x-y)^2}{2t}} f_0(y)\, dy$$

2.2 Tempered Distributions

Definition 2.27:

Let V be a topological vector space over a field $\mathbb{F}$ (here usually $\mathbb{F}=\mathbb{C}$).

Then the dual space V' is the space of all continuous linear maps $V \to \mathbb{F}$.

For $f \in V$, $T \in V'$ we write $\underbrace{T(f)}_{\in \mathbb{F}} = (f, T)_{V,V'}$ "natural pairing"

Note: • in finite dimensional vector spaces, elements in V (in some basis, column vector) can be identified with elements in V' (row vector)

• but in infinite dimensional spaces, V' is "larger" than V (dual to basis in V is not necessarily a basis)

Definition 2.26:

The elements of the dual space $S'(\mathbb{R}^d)$ of $S(\mathbb{R}^d)$ are called tempered distributions (or "generalized functions").

Ex.:

• for $(1+|x|^2)^{-m} g(x) \in L^1(\mathbb{R}^d)$ for some $m \in \mathbb{N}$, define

$$T_g : S(\mathbb{R}^d) \to \mathbb{C}, \; f \longmapsto \int g(x) f(x)\, dx$$

↳ T_g linear clear

↳ T_g continuous? If $f_n \xrightarrow{n\to\infty} f$ in S, does $T_g(f_n - f) \to 0$ (as a sequence in $\mathbb{C}$)?

$$|T_g(f_n-f)| = \left|\int g(x)\big(f_n(x)-f(x)\big)\,dx\right| \le \int |g(x)|\,|f_n(x)-f(x)|\,dx$$

$$\le \underbrace{\int (1+|x|^2)^{-m}|g(x)|\,dx}_{<\infty}\ \underbrace{\|(1+|x|^2)^m|f_n(x)-f(x)|\|_\infty}_{\xrightarrow{n\to\infty} 0}$$

$\Rightarrow T_g \in \mathcal{S}'$

- delta distribution $\delta: \mathcal{S} \to \mathbb{C},\ f \mapsto \delta(f) = f(0)$

$\Rightarrow \delta \in \mathcal{S}'$ clear $(|f_n(0)-f(0)| \le \|f_n-f\|_\infty)$

useful notation (in the spirit of previous example):

$\delta(f) = f(0) = \int \delta(x) f(x)\,dx$, and similarly $\int \delta(x-a)f(x)\,dx = f(a)$

but keep in mind that $\delta(x)$ is not a function $\mathbb{R}^d \to \mathbb{C}$!

however, it can be approximated by functions:

let $g \in L^1(\mathbb{R})$ $(d=1$ here$)$, $\int g(x)\,dx = 1$ and $g_n(x) = n\,g(nx)$ (a dilation as in HW 2)

s.t. $\int g_n(x)\,dx = \int n\,g(nx)\,dx = \int g(y)\,dy = 1$

$$\Rightarrow \lim_{n\to\infty} T_{g_n}(f) = \lim_{n\to\infty} \int g_n(x) \underbrace{f(x)}_{=f(0)+f(x)-f(0)}\,dx$$

$$= f(0) + \lim_{n\to\infty} \underbrace{\int n\,g(nx)\big(f(x)-f(0)\big)\,dx}_{= \int g(y)\underbrace{\left(f\left(\frac{y}{n}\right)-f(0)\right)}_{\xrightarrow{n\to\infty} 0 \text{ pointwise}}dy\ \xrightarrow{n\to\infty} 0 \text{ by dominated convergence}}$$

$$= f(0) = \delta(f)$$

We have two natural notions of convergence

<u>Definition 2.29:</u> V a topological vector space

a) $(f_n)_n$, $f_n \in V$ converges weakly to $f \in V$ if $\lim_{n\to\infty} T(f_n) = T(f)$ $\forall\, T \in V'$

notation: $\text{w-}\lim_{n\to\infty} f_n = f$ or $f_n \rightharpoonup f$

b) $(T_n)_n$, $T_n \in V'$ is a weak* convergent sequence with limit $T \in V'$ if

$$\lim_{n\to\infty} T_n(f) = T(f) \quad \forall\, f \in V$$

notation: $\text{w}^*\text{-}\lim_{n\to\infty} T_n = T$ or $T_n \xrightarrow{*} T$

<u>Ex.:</u> $T_{g_n} \xrightarrow{*} \delta$