Session 13
March 16, 2020

Recall: $\mathcal{H}$ a Hilbert space, $\{\varphi_j\}_j$ an ONS, then we have

- $\psi = \psi_n + \psi_n^\perp$, $\psi_n = \sum_{j=1}^n \langle \varphi_j, \psi \rangle \varphi_j$
- Bessel: $\|\psi\|^2 \geq \sum_{j=1}^n |\langle \varphi_j, \psi \rangle|^2$
- Cauchy-Schwarz: $|\langle \varphi, \psi \rangle| \leq \|\varphi\| \, \|\psi\|$
- polarisation identity: $\langle \varphi, \psi \rangle = \frac{1}{4}\left(\|\varphi+\psi\|^2 - \|\varphi-\psi\|^2\right) + \frac{i}{4}\left(\|\varphi - i\psi\|^2 - \|\varphi + i\psi\|^2\right)$

Next: some basic consequences of the concept of orthonormal basis

Definition 3.7: A sequence $(\varphi_j)_j$ in $\mathcal{H}$ is called orthonormal basis (ONB) if $\psi = \underbrace{\sum_{j=1}^\infty \langle \varphi_j, \psi \rangle \varphi_j}$ $\forall \psi \in \mathcal{H}$.

meaning $\|\psi - \sum_{j=1}^N \langle \varphi_j, \psi \rangle \varphi_j\|_{\mathcal{H}} \xrightarrow{N \to \infty} 0$

Note: With Zorn's lemma, every vector space has a basis (in the context of infinite dimensional $\mathbb{R}$ or $\mathbb{C}$ vector spaces called Hamel basis), meaning every vector can be written uniquely as a finite linear combination of basis vectors (recall Linear Algebra). Thus, ONBs are a different notion (but obviously it makes a lot of sense to also call them basis).

Consequences (here, $\mathcal{H}$ is a Hilbert space and $(\varphi_j)_j$ an ONB):

- The general definition of separability for topological spaces is the existence of a countable dense subset. By this definition, one can show that a Hilbert space is separable iff it has an ONB.

Proof: "$\Leftarrow$" $\left\{ \sum_{j=1}^N (a_j + i b_j) \varphi_j : N \in \mathbb{N}, a_j, b_j \in \mathbb{Q} \right\}$ is clearly a countable dense subset (of a complex Hilbert space)

"$\Rightarrow$" If $(\varphi_j)_j$ is any countable dense subset, we can – if necessary – just remove φ_j's such that the remaining $\{\varphi_j\}_{j \in J}$ are still $\underbrace{\text{linearly independent}}$, but still $\underbrace{\overline{\operatorname{span}\{\varphi_j\}_{j\in J}}} = \mathcal{H}$.

recall that for ∞-dim. vector spaces, this means all finite linear combinations are linearly independent

meaning $\psi = \sum_{j=1}^\infty a_j \varphi_j$ $\forall \psi \in \mathcal{H}$

The remaining basis can be made orthonormal by $\underbrace{\text{Gram-Schmidt}}_{\text{recall this from linear Algebra}}$. $\square$

Notes: • here, we are only interested in separable Hilbert spaces

• Examples of non-separable spaces:

↳ more from physics: infinite spin chain : $\underbrace{\bigotimes_{k\in\mathbb{Z}}}^{\text{tensor product (vigorously introduced later)}} \mathbb{C}^2$ (reason: think of the two basis vectors in $\mathbb{C}^2$ as 0 and 1, then basis vectors in the infinite tensor product are all 0,1 sequences; but there are as many such sequences as real numbers)

↳ ℓ^∞ (bounded real sequences) is a non-separable Banach space

↳ more from math: space of almost periodic functions $H=\overline{X}$ (completion of X), where $X=\{f:\mathbb{R}\to\mathbb{C},\ f(t)=\sum_{j=1}^{n} c_j e^{is_jt},\ s_j\in\mathbb{R}, c_j\in\mathbb{C}, n\in\mathbb{N}\}$ with scalar product

$$\langle f,g\rangle=\lim_{N\to\infty}\frac{1}{2N}\int_{-N}^{N}\overline{f(t)}\,g(t)\,dt$$ ($\{e^{ist}: s\in\mathbb{R}\}$ is an uncountable orthonormal set)

• Proposition 3.10: $\boxed{\text{An ONS } (\varphi_j)_j \text{ is an ONB iff: } \langle\varphi_j,\psi\rangle=0\ \forall j\in\mathbb{N} \Rightarrow \psi=0}$ (proof: HW 5)

• Parseval's identity: $\boxed{\|\psi\|^2=\sum_{j=1}^{\infty}|\langle\varphi_j,\psi\rangle|^2}$ ($\{\varphi_j\}$ ONB)

Proof: $\|\psi\|^2=\langle\lim_{N\to\infty}\sum_{j=1}^{N}\langle\varphi_j,\psi\rangle\varphi_j,\ \lim_{M\to\infty}\sum_{i=1}^{M}\langle\varphi_i,\psi\rangle\varphi_i\rangle$

$$\overset{\text{scalar product continuous}}{=}\lim_{N\to\infty}\lim_{M\to\infty}\underbrace{\langle\sum_{j=1}^{N}\langle\varphi_j,\psi\rangle\varphi_j,\sum_{i=1}^{M}\langle\varphi_i,\psi\rangle\varphi_i\rangle}_{\overset{\{\varphi_j\}\text{ ONB}}{=}\sum_{j=1}^{N}\overline{\langle\varphi_j,\psi\rangle}\langle\varphi_j,\psi\rangle}$$

$$=\lim_{N\to\infty}\sum_{j=1}^{N}|\langle\varphi_j,\psi\rangle|^2 \quad \square$$

- $\boxed{U: \mathcal{H} \to \ell^2,\ \psi \mapsto (\langle \varphi_j, \psi \rangle)_{j \in \mathbb{N}} \text{ is an isometric isomorphism}}$ (for separable $\mathcal{H}$) otherwise there would not be an ONB

<u>Proof:</u> Isometry due to Parseval $\|\psi\|_{\mathcal{H}}^2 = \sum_{j=1}^{\infty} |\langle \varphi_j, \psi \rangle|^2 = \|(\langle \varphi_j, \psi \rangle)_j\|_{\ell^2}^2 = \|U\psi\|_{\ell^2}^2$

injective clear bc. of isometry $\underbrace{\text{Isomorphism since } U \text{ surjective}}$: for any $(c_j)_{j \in \mathbb{N}} \in \ell^2$, we choose $\psi = \sum_{j=1}^{\infty} c_j \varphi_j$

$$\left(\psi \in \mathcal{H} \text{ since } \Big\|\sum_{j=N}^{\infty} c_j \varphi_j\Big\|_{\mathcal{H}}^2 = \Big\langle \sum_{j=N}^{\infty} c_j \varphi_j, \sum_{i=N}^{\infty} c_i \varphi_i \Big\rangle = \sum_{j=N}^{\infty} |c_j|^2 \xrightarrow{N \to \infty} 0 \ (c \in \ell^2) \right). \qquad \square$$

$\Rightarrow$ **All infinite dimensional separable Hilbert spaces are isometrically isomorphic to ℓ^2, and thus to each other.** (Finite dimensional Hilbert spaces are isometrically isomorphic to $\mathbb{C}^n$.)

↳ ℓ^2 is the coordinate space for any separable Hilbert space (of infinite dimension) (any choice of ONB gives us an isometric isomorphism)

Example: $(\varphi_k)_{k \in \mathbb{Z}}$ with $\varphi_k = (2\pi)^{-\frac{1}{2}} e^{ikx}$ is an ONB for $L^2([0, 2\pi])$;

$\psi = \sum_{k \in \mathbb{Z}} \langle \varphi_k, \psi \rangle \varphi_k$ is the Fourier series of ψ

Note: So why are we even interested in different Hilbert spaces? Because we are often interested in extra structure, e.g., operators on Hilbert spaces. Think of Fourier space, where differential operators become multiplication operators. (Or think of diagonalization in $\mathbb{C}^n$.)

<u>Definition 3.14:</u> For any $\overset{\text{any subset}}{\overbrace{M \subset \mathcal{H}}}$, we call $M^{\perp} := \{\psi \in \mathcal{H} : \langle \varphi, \psi \rangle = 0 \ \forall \varphi \in M\}$ the orthogonal complement of M.

Note: • $M \cap M^\perp = \begin{cases} \{0\} & \text{if } 0 \in M \\ \emptyset & \text{if } 0 \notin M \end{cases}$

• $M^\perp$ is a $\underbrace{\text{closed}}_{\langle \varphi, \cdot \rangle \text{ continuous}}$ $\underbrace{\text{subspace}}_{\langle \varphi, \cdot \rangle \text{ linear}}$ of $\mathcal{H}$

Theorem 3.15: Let $M \subset \mathcal{H}$ be a closed subspace. Then $\mathcal{H} = M \oplus M^\perp$, meaning $\forall \psi \in \mathcal{H}$ we have $\psi = \varphi + \varphi^\perp$ with unique $\varphi \in M, \varphi^\perp \in M^\perp$.

Proof: We give the proof only for separable $\mathcal{H}$.

$\Rightarrow$ also $M, M^\perp$ are separable Hilbert spaces with ONBs $(\varphi_i)_i$, $(\varphi_j^\perp)_j$.

Choose any $\psi \in \mathcal{H}$, def. $\varphi = \sum_{i=1}^{\infty} \langle \varphi_i, \psi \rangle \varphi_i$, $\varphi^\perp = \sum_{j=1}^{\infty} \langle \varphi_j^\perp, \psi \rangle \varphi_j^\perp$, then $\psi = \varphi + \varphi^\perp$ if $(\varphi_i)_i \cup (\varphi_j^\perp)_j$ is an ONB of $\mathcal{H}$.

Check with Proposition 3.10: Let $\underbrace{\langle \varphi_i, \psi \rangle = 0}$ $= \langle \varphi_j^\perp, \psi \rangle \ \forall i, j$

$\Rightarrow \langle \varphi, \psi \rangle = 0 \ \forall \varphi \in M \Rightarrow \psi \in M^\perp$

but since also $\langle \varphi^\perp, \psi \rangle = 0 \ \forall \varphi^\perp \in M^\perp \Rightarrow \psi = 0$

Uniqueness: suppose there is another decomposition $\psi = \underbrace{\tilde{\varphi}}_{\in M} + \underbrace{\tilde{\varphi}^\perp}_{\in M^\perp}$

$\Rightarrow \varphi + \varphi^\perp = \tilde{\varphi} + \tilde{\varphi}^\perp$, i.e., $M \ni \varphi - \tilde{\varphi} = \tilde{\varphi}^\perp - \varphi^\perp \in M^\perp$

but $M \cap M^\perp = \{0\}$ so $\varphi = \tilde{\varphi}$ and $\varphi^\perp = \tilde{\varphi}^\perp$. $\square$