We continue or study of self-adjointness.

Last time we studied closedness and its relation to adjoints. With the properties we have established we can now introduce another "nice" class of unbounded operators:

Definition 3.71: A densly defined symmetric operator is called essentially self-adjoint if its closure is self-adjoint.

Why is this nice? Because:

Corollary 3.72: A densly defined symmetric operator T is essentially self-adjoint if and only if T^* is symmetric. In that case $\overline{T} = T^*$ and $\overline{T}$ is the unique self-adjoint extension of T.

So for essentially self-adjoint operators we have a unique self-adjoint extension $\overline{T}$. In many applications it is enough to know that. (Computing $\overline{T}$ explicitly can be hard or easy.)

Proof:

by Corollary 3.70: $\overline{T} \subset T^*$ and $T^* = T^{***}$ and $\overline{T} = T^{**}$

- if also T^* symmetric, then Corollary 3.70 a) applied to T^* tell us that $T^{***} \subset T^{**}$, i.e., $T^* \subset \overline{T}$. So, $T^* = (\overline{T})^* = \overline{T}$.

- uniqueness of extension: Let S be another self-adjoint extension. Then
$T \subset S \Rightarrow \overline{T} \subset \overline{S} = S$ (S self-adjoint $\Rightarrow S$ closed), and with Proposition 3.67 we have

$S = S^* \subset T^* = \overline{T}$, so $S = \overline{T}$.

Example: $D_{min} = -i\frac{d}{dx}$ with $D(D_{min}) = \{\psi \in H^1([0,1]) : \psi(1) = 0 = \psi(0)\}$ is symmetric, but not essentially self-adjoint, since it has many self-adjoint extensions.

On the other hand, e.g., $(-\Delta, C_0^\infty(\mathbb{R}^d))$ should be essentially self-adjoint. In fact, it seems like there are many other choices of domains on which $-\Delta$ is essentially self-adjoint. So let us define the following:

<u>Definition 3.73:</u> Let $(T, D(T))$ be self-adjoint. A subspace $D_0 \subset D(T)$ that is dense in $\mathcal{H}$ is called core (or essential domain) of T if (T, D_0) is essentially self-adjoint, i.e., $\overline{T|_{D_0}} = T$.

In other words, D_0 is a core of $(T, D(T))$ if D_0 is dense in $D(T)$ w.r.t. the graph norm $\|\varphi\|_T^2 := \|T\varphi\|_{\mathcal{H}}^2 + \|\varphi\|_{\mathcal{H}}^2$.

Let us discuss some examples:

- Multiplication operators: Let $V : \mathbb{R}^d \to \mathbb{C}$ be measurable and $M_V : D(M_V) \to L^2(\mathbb{R}^d)$, $f \mapsto M_V f$ with $(M_V f)(x) = V(x) f(x)$, with domain $D(M_V) = \{f \in L^2(\mathbb{R}^d) : Vf \in L^2(\mathbb{R}^d)\}$ dense in $L^2(\mathbb{R}^d)$.

  Then the adjoint M_V^* is given by $(M_V^* f)(x) = \overline{V(x)} f(x)$ and has domain $D(M_V^*) = D(M_V)$.

  So if V is real, M_V on its maximal domain $D(M_V)$ is self-adjoint.

- Laplace operator: With the example above we can deduce that $H_0 = -\Delta$ with $D(H_0) = H^2(\mathbb{R}^d)$ is self-adjoint, since $\mathcal{F} H_0 \mathcal{F}^{-1} = |k|^2$ as multiplication operator with its maximal domain is self-adjoint, and since we have the following lemma:

Lemma 3.76: Let $U: \mathcal{H}_1 \to \mathcal{H}_2$ be unitary and $(H, D(H))$ self-adjoint on $\mathcal{H}_1$. Then $(UHU^*, UD(H))$ is self-adjoint on $\mathcal{H}_2$.

Proof: HW

But furthermore, any subspace $X \subset H^2(\mathbb{R}^d)$ which is dense w.r.t. the graph norm of H_0, i.e., the H^2-norm $\|\varphi\|_{H^2}^2 = \|\Delta\varphi\|_{L^2}^2 + \|\varphi\|_{L^2}$, is a core for H_0. So, e.g., $C_0^\infty(\mathbb{R}^d)$ is a core for $(H_0, H^2(\mathbb{R}^d))$, i.e., $(H_0, C_0^\infty(\mathbb{R}^d))$ is essentially self-adjoint.

On the other hand $X_0 = C_0^\infty(\mathbb{R}^d \setminus \{0\})$ for $d \le 3$ is not a core for $(H_0, H^2(\mathbb{R}^d))$, because X_0 is not dense in $H^2(\mathbb{R}^d)$ (it is dense in $L^2(\mathbb{R}^d)$!). This leaves room for many self-adjoint extensions, corresponding to "δ potentials" or different boundary conditions at the origin. (We will come back to this later.)