Jacobs University
Spring 2018

May 28, 2018

Foundations of Mathematical Physics

Final Exam

Instructions:

- Do all the work on this exam paper.
- Show your work, i.e., carefully write down the steps of your solution.
- Calculators and other electronic devices and notes are not allowed.
- You are free to refer to any results proven in class or the homework sheets unless otherwise stated (and unless the problem is to reproduce a result from class or the homework sheets).

Name: ____________________________________

Problem 1: Fourier Transform and Schwartz Functions [30 points]

Let $\mathcal{F}(f)(k) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-ikx}\, dx$ denote the Fourier transform of a function f.

(a) Let $f \in L^1(\mathbb{R}^d)$. Does this imply that $\mathcal{F}(f)(k)$ is continuous? Briefly explain your answer. (*Hint: Recall the lemma about integrals with parameters from class.*)

(b) Define the space of Schwartz functions $\mathcal{S}(\mathbb{R}^d)$. What does it mean for a sequence of functions $f_n \in \mathcal{S}(\mathbb{R}^d)$ to converge to $f \in \mathcal{S}(\mathbb{R}^d)$?

(c) Let $\psi_0 \in \mathcal{S}(\mathbb{R}^d)$ and consider the solution to the free Schrödinger equation

$$\psi : \mathbb{R}_t \to \mathcal{S}(\mathbb{R}^d), t \mapsto \psi(t) = \mathcal{F}^{-1} e^{-i\frac{k^2}{2}t} \mathcal{F}\psi_0.$$

Prove that this map is differentiable.

(d) Let $\mathcal{S}'(\mathbb{R}^d)$ be the space of tempered distributions, i.e., the dual space of $\mathcal{S}(\mathbb{R}^d)$. For $d = 1$, we define

$$T_\Theta : \mathcal{S}(\mathbb{R}) \to \mathbb{C}, f \mapsto \int_{\mathbb{R}} \Theta(x) f(x)\, dx, \quad \text{with } \Theta(x) = \begin{cases} 1 & ,\text{for } x \geq 0 \\ 0 & ,\text{for } x < 0. \end{cases}$$

Prove that $T_\Theta \in \mathcal{S}'(\mathbb{R})$, i.e., that T_Θ is linear and continuous.

(e) For any multi-index $\alpha \in \mathbb{N}_0^d$ and for $T \in \mathcal{S}'(\mathbb{R}^d)$, how is the distributional derivative $\partial_x^\alpha T$ defined? Compute the distributional derivative of T_Θ.

Problem 1: Extra Space

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Problem 2: Unitary Groups [30 points]

(a) Define what a unitary operator between two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ is.

(b) Recall that a densely defined linear operator H with domain $D(H) \subset \mathcal{H}$ is called a generator of a strongly continuous one-parameter group $U(t)$ if

$$D(H) = \big\{\psi \in \mathcal{H} : t \mapsto U(t)\psi \text{ is differentiable}\big\} \quad \text{and} \quad i\frac{d}{dt}U(t)\psi = U(t)H\psi.$$

Now let H be a generator of $U(t)$. Prove that

(i) $U(t)D(H) = D(H)$ for all t,

(ii) $HU(t)\psi = U(t)H\psi$ for all $\psi \in D(H)$,

(iii) $\langle H\psi, \varphi\rangle = \langle \psi, H\varphi\rangle$ for all, $\varphi, \psi \in D(H)$.

(c) Define the notions of weak and strong convergence for bounded operators on some Hilbert space $\mathcal{H}$.

(d) One can show that $(T_a(t))_t$ which is for all $a \in \mathbb{R}$ defined by $(T_a(t)\psi)(x) := \psi(x - at)$ is indeed a strongly continuous unitary group on $L^2(\mathbb{R})$. Prove that $T_a(t)$ converges weakly but not strongly to 0 as $t \to \infty$.

Problem 2: Extra Space

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Problem 3: Self-adjoint Operators [30 points]

(a) State the Riesz Representation Theorem for a Hilbert space $\mathcal{H}$ and an element $T \in \mathcal{H}'$ in its dual space.

(b) Let A be a bounded operator. Define what it means for A to be self-adjoint. How is this related to A being symmetric? Is A a generator of the strongly continuous unitary group e^{-iAt}? (No proofs necessary here.)

(c) Let T be an unbounded operator with domain $D(T)$. How is the adjoint of T defined and what is its domain? Define what T self-adjoint and T essentially self-adjoint mean.

(d) Recall from class that a densely defined symmetric operator H with domain $D(H)$ is essentially self-adjoint if and only if $\ker(H^* \pm i) = \{0\}$. Use this criterion to determine whether

(i) $D_{\min} = -i\frac{d}{dx}$ with domain $D(D_{\min}) = \{\psi \in H^1([0,1]) : \psi(1) = \psi(0) = 0\}$,

(ii) $D_{[0,\infty)} = -i\frac{d}{dx}$ with domain $D(D_{[0,\infty)}) = C_0^\infty((0,\infty))$,

(iii) $H_0 = -\Delta$ with domain $D(H_0) = C_0^\infty(\mathbb{R}^d)$

are essentially self-adjoint. For those that are, determine on which domain they are self-adjoint. For those that are not, determine whether self-adjoint extensions exist by studying the deficiency indices.

Problem 3: Extra Space

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Problem 4: Spectrum and Spectral Theorem [30 points]

(a) Let $(T, D(T))$ be a linear operator on some Hilbert space $\mathcal{H}$. Define what the resolvent $R_z(T)$, the resolvent set $\rho(T)$ and the spectrum $\sigma(T)$ of $(T, D(T))$ are.

(b) Let $(A, D(A))$ be a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$ and let $(A_n, D(A_n))$ be a sequence of bounded self-adjoint operators on $\mathcal{H}$ such that for some $C > 0$, the $\sup_{n\in\mathbb{N}} \|A_n\|_{\mathcal{L}(\mathcal{H})} \leq C$. Prove that convergence of A_n to A in operator norm implies that there is a $z \in \mathbb{C} \setminus \mathbb{R}$ such that

$$\lim_{n\to\infty} \|R_z(A_n) - R_z(A)\|_{\mathcal{L}(\mathcal{H})} = 0.$$

(Here, $\|\cdot\|_{\mathcal{L}(\mathcal{H})}$ denotes the operator norm.)

(c) Briefly and informally state the three different versions of the spectral theorem for unbounded self-adjoint operators that were discussed in class.

(d) Briefly summarize how we defined in class a functional calculus for some nice class of functions $f : \mathbb{R} \to \mathbb{C}$ with the help of the Helffer-Sjöstrand formula.

Problem 4: Extra Space

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